MATH 691, Spring 2002
Midterm Exam

25th February 2002

1. Suppose that we wish to partition an undirected graph into two components with the minimum number of arcs between the components (i.e. arcs with one endpoint in each of the two components). How would you solve this problem by solving a single maximum flow problem? You should clearly define the network in which you would find the maximum flow (including network topology and arc capacities) and which nodes will be considered the source and the sink. Your network must satisfy all assumptions that we discussed for the maximum flow problem. You should also clearly explain why solving the maximum flow problem in your graph will yield a graph partition as described.

2. When discussing maximum flow problems, we assume that the network has no $s-t$ path with infinite capacity. Suppose you are given network $G = (N, A)$. Give an $O(m)$ procedure that identifies whether or not $G$ satisfies this assumption. You should clearly outline your procedure, analyze its run-time, and show that it is correct (i.e. achieves the desired result).

3. Prove or disprove the following statement:
   Let $G = (N, A)$ be a network with source node $s \in N$ in which each arc $(i, j)$ has associated cost $c_{ij}$. Let $(k, l)$ be the edge of minimum cost in $G$; that is, $c_{kl} \leq c_{ij} \forall (i, j) \in A$. Then there is some node $p$ whose shortest path from $s$ contains arc $(k, l)$.

4. Suppose you are given a network $G = (N, A)$ with capacity $u_{ij}$ on arc $(i, j)$ for each $(i, j) \in A$. Suppose that this network has several source nodes $(s_1, s_2, \ldots, s_p)$, and several sink nodes $(t_1, t_2, \ldots, t_q)$. Any unit of flow can originate in any node and may terminate in any sink. The goal is to find the maximum total flow. Show how to transform this
problem into a maximum flow problem with only one source node and only one sink node. You should clearly define the resulting network (topology and capacity structure) and which nodes will be considered the source and the sink. Your network must satisfy all assumptions that we discussed for the maximum flow problem. You should also clearly explain why solving the maximum flow problem in your single-source, single-sink network will correspond to a maximum total flow in the multi-source, multi-sink problem.