§12.7, problem 4

The model we must solve is

\[
\begin{align*}
\text{max} & \quad 30x_1^{\frac{4}{3}} + 20x_2^{\frac{4}{3}} - x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 = 100.
\end{align*}
\]

We begin by forming the Lagrangian, which is

\[
L = 30x_1^{\frac{4}{3}} + 20x_2^{\frac{4}{3}} + \lambda(100 - x_1 - x_2) - x_1 - x_2. 
\]

Then the partial derivatives with respect to \(x_1, x_2, \) and \(\lambda\) are

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 15x_1^{-\frac{1}{3}} - \lambda - 1 \\
\frac{\partial L}{\partial x_2} &= 10x_2^{-\frac{1}{3}} - \lambda - 1 \\
\frac{\partial L}{\partial \lambda} &= 100 - x_1 - x_2. 
\end{align*}
\]

Now we must find the stationary points; that is, all points where

\[
\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial \lambda} = 0.
\]

Using \(\frac{\partial L}{\partial x_1} = 0,\) we obtain \(x_1 = \frac{225}{(\lambda+1)^{\frac{2}{3}}}.\)

Using \(\frac{\partial L}{\partial x_2} = 0,\) we obtain \(x_2 = \frac{100}{(\lambda+1)^{\frac{2}{3}}}.\)

Then \(\frac{\partial L}{\partial \lambda} = 0,\) we obtain \((\lambda + 1)^2 = 3.25 \implies \lambda = 0.8.\)

Using substitution, then, we obtain \(x_1 = 69.23\) and \(x_2 = 30.77.\) Thus
the beer company should spend $69.23 on promotion in territory 1, and $30.77 on promotion in territory 2. If they have an extra dollar to spend on promotion, then that will result in an increase of 80 cents in profit.

- §12.8, problem 2 The problem is

\[
\begin{align*}
\text{max} & \quad x_1 - x_2 \\
\text{s.t.} & \quad x_1^2 + x_2^2 \leq 1.
\end{align*}
\]

The Kuhn-Tucker conditions given in Theorem 9 imply

\[
\begin{align*}
1 - \lambda (2x_1) &= 0 \\
-1 - \lambda (2x_2) &= 0 \\
\lambda (1 - x_1^2 - x_2^2) &= 0
\end{align*}
\]

Using the first equation, we have that \(x_1 = \frac{1}{2\lambda}\), and using the second equation, we have that \(x_2 = -\frac{1}{2\lambda}\). Plugging these into the third equation, we get

\[
\begin{align*}
\lambda (1 - \left(\frac{1}{2\lambda}\right)^2 - \left(-\frac{1}{2\lambda}\right)^2) &= 0 \\
\Rightarrow \lambda (1 - \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2}) &= 0 \\
\Rightarrow \lambda (1 - \frac{1}{2\lambda^2}) &= 0 \\
\Rightarrow \lambda - \frac{1}{2\lambda} &= 0 \\
\Rightarrow \lambda &= \frac{1}{\sqrt{2}}
\end{align*}
\]

Notice that \(\lambda \geq 0\). Using this value of \(\lambda\), then, we obtain \(x_1 = \frac{1}{\sqrt{2}}\) and \(x_2 = -\frac{1}{\sqrt{2}}\).

- §12.8, problem 3

The LP we are concerned with is

\[
\begin{align*}
\text{max} & \quad 3x_1 + x_2 \\
\text{s.t.} & \quad 2x_1 + x_2 \leq 100 \\
& \quad x_1 + x_2 \leq 80 \\
& \quad x_1 \leq 40 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
The (modified) Kuhn-Tucker conditions given on p. 694 in your text, and which are numbered (30r)–(33r) yield the following conditions for this problem:

\[
\begin{align*}
3 - 2\lambda_1 - \lambda_2 - \lambda_3 \leq 0 \\
2 - \lambda_1 - \lambda_2 \leq 0 \\
\lambda_1(100 - 2x_1 - x_2) = 0 \\
\lambda_2(80 - x_1 - x_2) = 0 \\
\lambda_3(40 - x_1) = 0 \\
(3 - 2\lambda_1 - \lambda_2 - \lambda_3)x_1 = 0 \\
(2 - \lambda_1 - \lambda_2)x_2 = 0 \\
\lambda_1, \lambda_2, \lambda_3 \geq 0
\end{align*}
\]

Notice that (1), (2), and (8) together give the dual of the original LP (minus an objective). Conditions (3)–(5) are the complementary slackness conditions stating that either the primal constraint is tight or the corresponding dual variable is zero. Conditions (6)–(7) are the complementary slackness conditions stating that either the dual constraint is tight or the corresponding primal variable is zero.