Section 11

1. Do you think two or more students will arrive late?

2. How did you plan to deal with a situation where you were asked to do something?

3. What did you learn about the way students react to being asked to do something?

4. How did you find out whether students were willing to do something?
Section 1.2

1.2 Linear Functions

1.4 Power Functions

1.10 Polynomials and Rational Functions

1.9 Trig Functions

Brief thing on exponential functions + log functions. Show how exponential tends to power functions.

1.8 New Functions from Old
- shifts, translations, reflections
- sum
- composition
- odd/even again?
- show these w/ trig
  fps. or polyn...
I will give a series of around four lectures on this section, adding some of my own material and not covering some material in the text. You are responsible for the additions and what I do not cover in the text.

Outline:
1. Linear Functions
2. Power Functions
3. Polynomial and Rational Functions
4. Trig Functions
5. Briefly, Exponential and Log Functions
6. New Functions from Old
    - Shifts, Compositions, Etc.
Lecture

LINEAR FUNCTIONS.

Recall from precalculus:

1) **Slope-intercept equation of a line**

\[ y = mx + b \]

slopes \( m = \frac{\text{rise}}{\text{run}} \)

2) **Point-slope equation of a line**

\[ y - y_1 = m(x - x_1) \]

slope \( m = \frac{y_2 - y_1}{x_2 - x_1} \)

particular point \((x_1, y_1)\) on the line

3) **Distance formula for a line segment**

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
A linear function has the form
\[ f(x) = mx + b \], \( m \) and \( b \) constants,
where
\[ m = \text{slope of graph of } f \]
\[ b = \text{y-intercept of graph of } f \]

E.g., \( f(x) = 2x + 1 \)

Graph of \( f \):

\[ y = 2x + 1 \]

\[ (1, 3) \]

Lecture
Note the following:

\[ f(x) = mx + b \]

Graph of \( f \):

\[ y = f(x) = mx + b \]

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]

DIFERENCE QUOTIENT

(will come up in Ch. 2)
Properties of Lines

1) Increasing (Positive Slope)
   
   \[ y = x + 1 \]

2) Decreasing (Negative Slope)
   
   \[ y = -x + 1 \]

3) Parallel Lines
   
   \[ y = 3x + 5 \] is parallel to \[ y = 3x - 1 \]

\[ m_1 = m_2 \]
4) Perpendicular Lines

\[ m_1 = -\frac{1}{m_2} \text{ or } m_2 = -\frac{1}{m_1} \]

E.g., \( y = \frac{1}{2}x + 1 \) is perp. to \( y = -3x + 7 \)

5) Horizontal (Zero Slope)

E.g., \( y = 2 \)

6) Vertical (Undefined or Infinite Slope)

E.g., \( x = 1 \)
Example. Find the equation of the line that passes through the points \((0, 2)\) and \((2, 3)\)

\[ y - y_1 = m(x - x_1) \]

\[ \begin{align*}
&1. m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 2}{2 - 0} = \frac{1}{2} \\
&2. (x_1, y_1) = (0, 2) \\
&3. y - 2 = \frac{1}{2}(x - 0) \Rightarrow y = \frac{1}{2}x + 2
\end{align*} \]

[You will use some of this material later in Ch. 2.]
POWER FUNCTIONS.

A power function has the form

\[ f(x) = kx^r \]

where \( k \) and \( r \) are constants.

Note that \( r \) can be any real number, e.g.,

\[ f(x) = \frac{1}{3}x^3, \quad f(x) = \sqrt{x}, \quad f(x) = x^{5/4}, \quad f(x) = 2x^3 \]

The Graphs of Power Functions

1. \( r \) is odd or even positive integer

\[ \begin{align*}
A & \quad \text{"U"} \\
E.g., & \quad f(x) = x^2 \\
& \quad f(x) = x^4 \\
& \quad f(x) = x^6 \\
\end{align*} \]

\[ \begin{align*}
A & \quad \text{"sew"} \\
E.g., & \quad f(x) = x^2 \\
& \quad f(x) = x^5 \\
& \quad f(x) = x^7 \\
\end{align*} \]
2. \( r = 0, 1, \) or \(-1\)

- **Horizontal Line**: \( y = x^0 = 1 \)

- **Diagonal Line**: \( y = x^1 = x \)

- **Hyperbola**: \( y = x^{-1} = \frac{1}{x} \)

3. \( r = \) odd or even negative integer (includes \( r = -1 \))

E.g., \( f(x) = x^{-2} \)
- \( f(x) = x^{-4} \)
- \( f(x) = x^{-6} \)

E.g., \( f(x) = x^{-1} \)
- \( f(x) = x^{-3} \)
- \( f(x) = x^{-5} \)
4. \( r = \text{positive fraction} \)

\[ f(x) = x^r \quad (r > 0) \]

E.g., \( f(x) = x^{1/2} = \sqrt{x} \)

E.g., \( f(x) = x^{1/3} = \sqrt[3]{x} \)

\[ f(x) = x^r \]

You will use some of this material later in Ch. 3.
Lecture

POLYNOMIAL FUNCTIONS

The general formula for a polynomial function is this:

\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

- All exponents are nonnegative integers (i.e., positive integers and zero)
- \( n \) = degree of \( p(x) \)
- \( a_0, a_1, a_2, \ldots, a_{n-1}, a_n \) = coefficients - real constants
- \( a_n \) = leading coefficient

E.g., \( p(x) = -5x^{10} + 2x^9 + \frac{1}{4}x^7 + x^6 - 11x^3 - 120x + 7 \)

- Exponents are 0, 1, 2, 6, 7, 9, 10
- 10 = degree of \( p(x) \)
- 7, -120, -11, 1, \( \frac{1}{4} \), 2, -5 = coefficients
- -5 = leading coefficient

Examples of functions that are not polynomial functions:

1. \( f(x) = x^2 + 2x + 1 + \frac{1}{x} \)
   - \( x < -1 \)
   - Negative integer
2. \( f(x) = 3x^4 + \sqrt{x^3} \)  
\[ \frac{\sqrt{x^3}}{x^2} \]  
not an integer

3. \( f(x) = \sqrt[4]{x^2 + x^2 + x + 1} \)

radical over a polynomial expression

---

Graphs (and Their "Bumps")

Example:

\[ f(x) = \sqrt{x^2 - 1} \]

one "valley"

\[ f(x) = \sqrt[3]{x^3 - 2x^2 - x + 2} \]

one "peak," one "valley"

\[ f(x) = \sqrt{x^2 - 5x^2 + 5x^2 + 5x} \]

two "valleys," one "peak"
If degree \( p(x) = n \), then the graph of \( p(x) \) has at most \( n - 1 \) "bumps" (= total number of "peaks" and "valleys")

Some common types of polynomial functions:
- Linear function: \( p(x) = ax + b \)
- Quadratic function: \( p(x) = ax^2 + bx + c \)
- Cubic function: \( p(x) = ax^3 + bx^2 + cx + d \)

Graphing polynomials in factored form:

There is a theorem that says if \( p(x) \) is a polynomial of degree \( n \),

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,
\]

then it can be factored into the product of \( n \) factors.
\[ p(x) = k(x - c_n)(x - c_{n-1}) \cdots (x - c_1) \]

- \( k = a_n \)
- \( c_1, \ldots, c_{n-1}, c_n \) = real or complex numbers

Once in factored form, \( p(x) \) generally can be easily graphed.

E.g., \( p(x) = x^3 - 3x - 2 \)

\( p(x) \) can be written in factored form:

\[ p(x) = (x - 2)(x + 1)^2 \] \[ = (x - 2)(x + 1)(x + 1) \]

The solutions of the equation \( p(x) = 0 \):

\[ (x - 2)(x + 1)^2 = 0 \]

are called the zeros of \( p(x) \):

Zeros: \( x = 2, -1, -1 \)

2 = zero of multiplicity one

1 = zero of multiplicity two

\( p(x) \) crosses the \( x \)-axis at zeros of multiplicity one, and touches the \( x \)-axis at zeros of multiplicity two;
x-intercepts:

\[ x = 2 \quad \text{cross} \]
\[ x = -1 \quad \text{touch} \]

Graph:

\[ p(-2) = -4 < 0 \Rightarrow \text{graph below} \]
\[ p(2) = 20 > 0 \Rightarrow \text{graph above} \]

Sketch of graph of \( p(x) \) is:

\[ y = x^3 - 3x - 2 \]

[You will use some of this material later in Ch. 2.]
Lecture

RATIONAL FUNCTIONS

Rational functions are of the form

\[ f(x) = \frac{p(x)}{q(x)} \]

where

- \( p(x) = \) polynomial function
- \( q(x) = \) polynomial function
- can be equal to 0 some (but not all) of the time
- when \( q(x) = 0 \), \( f(x) \) is undefined

---

Graphing a Rational Function

**Example.** Sketch the graph of the function

\[ f(x) = \frac{1-4x}{2x+2} \]

**STEP 1.** Rewrite as

\[ f(x) = \frac{-4x+1}{2x+2} \]

Write polys such that their exponents are in descending order
STEP 2. Vertical Asymptotes? (vertical lines that the graph cannot cross but gets closer and closer to)

Set denominator = 0 and solve for $x$: $2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$.

:. Equation of vertical asymptote is $\boxed{x = -1}$

STEP 3. Horizontal Asymptote? (horizontal line that the graph cannot cross but gets closer to)

Look at $f(x)$ for large $|x|$. For large $|x|$, $y = f(x) = \frac{-4x + 1}{2x + 2} \approx \frac{-4x}{2x} = \frac{-4}{2} = -2$.

For large $|x|$, 1 becomes negligible compared to $-4x$ and 2 becomes negligible compared to $2x$.

:. Equation of horizontal asymptote is $\boxed{y = -2}$.
STEP 4. *x*-intercepts? (= where graph crosses/ touches *x*-axis)

Set numerator = 0 and solve for *x* ⇒

\[-4x + 1 = 0 \Rightarrow -4x = -1 \Rightarrow x = \frac{1}{4}\]

, 'x'-intercept is \(x = \frac{1}{4}\) or

\[\left(\frac{1}{4}, 0\right)\]

STEPS. *y*-intercept? (= where graph crosses *y*-axis)

Set *x* = 0 and compute \(y = f(x)\) ⇒

\[y = f(0) = \frac{-4(0) + 1}{2(0) + 2} = \frac{1}{2}\]

, 'y'-intercept is \(y = \frac{1}{2}\) or

\[\left(0, \frac{1}{2}\right)\]
**STEP 6.** Determine sample points before and after vertical asymptote:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = \frac{-4x+1}{2x+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-9/2 = -4.5</td>
</tr>
<tr>
<td>-1</td>
<td>undefined</td>
</tr>
<tr>
<td>1</td>
<td>-3/4 = -0.75</td>
</tr>
</tbody>
</table>

**STEP 7.** Sketch asymptotes and "important" points:

![Graph with asymptotes and points](image-url)

- Asymptote $y = -2$
- Point $(1, -0.75)$
- Point $(-2, -4.5)$
STEP 8. Draw curves through "important" points such that they fit into the corners formed from the intersection of the asymptotes.

You will use some of this material later in Ch. 2.
TRIGONOMETRIC FUNCTIONS

These are functions whose independent variable \( t \) is an angle.

\[
y = f(t)
\]

One measures angles in degrees or radians; radians are preferred in calculus:

\[
180^\circ \sim \pi \text{ (radians)} \Rightarrow \text{equivalent to}
\]

1) \( \# \text{ radians} = \# \text{ degrees} \times \frac{\pi}{180^\circ} \)

2) \( \# \text{ degrees} = \# \text{ radians} \times \frac{180^\circ}{\pi} \)
Illustration of Angles in Cartesian Plane

\[ \frac{2\pi}{3} (120^\circ) \]

\[ \frac{\pi}{2} (90^\circ) \]

\[ \frac{\pi}{3} (60^\circ) \]

\[ \frac{\pi}{4} (45^\circ) \]

\[ \frac{\pi}{6} (30^\circ) \]

\[ -\frac{\pi}{2} (-90^\circ) \]

\[ \pi (180^\circ) \]

\[ \frac{3\pi}{2} (270^\circ) \]

\[ 0 (0^\circ) \]

\[ 2\pi (360^\circ) \]
Two Fundamental Trig Functions

\[ f(t) = \sin t \quad \quad f(t) = \cos t \]

They are defined with respect to the unit circle (radius = 1, center = origin):

\[ x = \cos t \quad \quad y = \sin t \]

By the Pythagorean theorem,

\[ x^2 + y^2 = 1 \Rightarrow \]
\[ \cos^2 t + \sin^2 t = 1 \]

**FUNDAMENTAL (TRIG) IDENTITY**

Also, there is the tangent function:

\[ \tan t = \frac{\sin t}{\cos t} \]

Slope = \[ \frac{\sin t}{\cos t} = \tan t \]
Graphs of $\sin t$, $\cos t$, and $\tan t$

$y = \sin t$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$y = \sin t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>-1</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
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</table>

$y = \cos t$

<table>
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<td>-1</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>0</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1</td>
</tr>
</tbody>
</table>

Amplitude $= \frac{1}{2} (\max - \min)$
$= \frac{1}{2} (1 - (-1))$
$= 1$

Period $= \text{the part of the graph that repeats itself}$
$= 2\pi$

Amplitude $= 1$

Period $= 2\pi$
\[ y = \tan t \]

(Amplitude = \infty)

Period = \pi

Undefined where \cos t = 0:
\[ t = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \]
# How to Determine Amplitude and Period From Formula

<table>
<thead>
<tr>
<th>Formula</th>
<th>Amplitude</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) = a \sin (bt) )</td>
<td>( a )</td>
<td>( \frac{2\pi}{b} )</td>
</tr>
<tr>
<td>( f(t) = \cos (bt) )</td>
<td>( a )</td>
<td>( \frac{2\pi}{b} )</td>
</tr>
<tr>
<td>( f(t) = \tan (bt) )</td>
<td>undefined</td>
<td>( \frac{\pi}{b} )</td>
</tr>
</tbody>
</table>
Lecture

**EXPONENTIAL FUNCTIONS (Brief Review)**

Exponential functions are of the form

\[ f(x) = ka^x \]

where \( k \) and \( a \) are constants, \( a > 0 \), \( a \neq 1 \), and \( x \) is real.

**Exponential Functions vs. Power Functions**

- **Exponent is constant**
- **Base is variable**
- **Exponent is variable**
- **Base is constant**

**Graphs of Exponential Functions**

- **Increasing**
- **Horizontal asymptote: x-axis**
- **Y-intercept: \( y = 1 \)**

- **Decreasing**
- **Horizontal asymptote: x-axis**
- **Y-intercept: \( y = 1 \)**

1/25/01
Tues.
Lecture

1/24/01
Wed.
Lecture
E.g., \( f(x) = 2^x \) \hspace{1cm} \( f(x) = \left(\frac{1}{2}\right)^x \)
Lecture

LOGARITHMIC FUNCTIONS (Brief Review)

Logarithmic functions are of the form

\[ f(x) = \log_a x \]

\[ a \text{ a constant} \]

\[ a > 0, \ a \neq 1 \]

\[ x > 0 \]

Log functions are the "inverse functions" of exponential functions (SEE Sect. 1.6), and so their graphs are the reflections of the graphs of exponential functions about the diagonal line, \( y = x \):

\[ f(x) = \log_a x, \ a > 1 \]

\[ f(x) = \log_a x, \ 0 < a < 1 \]

- Increasing
- Vert. Asympt.: y-axis
- x-int. = 1

- Decreasing
- Vert. Asympt.: y-axis
- x-int. = 1
E.g., \( f(x) = \log_2 x \)  \hspace{1cm} f(x) = \log_\frac{1}{2} x
NEW FUNCTIONS FROM OLD.

There are many modifications one can make with functions resulting in new functions with a new graph.

Change in shape:
E.g., \( y = x^2 \) \( \rightarrow \) \( y = \frac{1}{2} x^2 \)

Change in location:
E.g., \( y = x^2 \) \( \rightarrow \) \( y - 2 = (x-1)^2 \)
Change in orientation:

E.g., $y = x^2$  $y = -x^2$

We will now be more specific mathematically.

1. **Multiple of a function:** $y = kf(x)$

E.g., $y = x^2$  $y = \frac{1}{2}x^2$

compress vertically
2. *Multiple of x*: \( y = f(kx) \)

*Example:* \( y = \sin x \)  
\( y = \sin \left( \frac{1}{2}x \right) \)

- **Period:** \( 2\pi \)  
- **Period:** \( \frac{2\pi}{k} = \frac{2\pi}{1/2} = 4\pi \)

- **Stretch horizontally**
- **Compress horizontally**

*Reflect across the y-axis* (in this case, also reflection across x-axis since \( \sin(-x) = -\sin x \))
3. **Shifts of a Function:** \( y - k = f(x - h) \)  
(or \( y = f(x - h) + k \))

E.g., \( y = x^2 \)  
\[
\begin{align*}
\text{vertical shift up} & : y + 4 = x^2 \\
& \quad (y = x^2 + 4) \\
\text{vertical shift down} & : y - 4 = x^2 \\
& \quad (y = x^2 - 4)
\end{align*}
\]

\( y = (x - 2)^2 \)  
\[
\begin{align*}
\text{horizontal shift to the right} & : x + 2 = y \\
\end{align*}
\]
$y = (x + 2)^2$

Horizontal shift to the left.
4. **Sum of two functions**: \((f+g)(x) = f(x) + g(x)\)

E.g., \(f(x) = 2^x\), \(g(x) = 2^{-x}\) \(\Rightarrow\) 
\[(f+g)(x) = 2^x + 2^{-x}\]

\[y = 2^x + 2^{-x}\]

---

5. **Composition of two functions**: 
\[
(f \circ g)(x) = f(g(x))
\]
Composed \(f\) of \(g\) of \(x\)

E.g., \(f(x) = \ln(x+3)\), \(g(x) = e^{4x+7}\)
\[
\begin{align*}
f(g(x)) &= f(e^{4x+4}) = \frac{\ln(e^{4x+4} + 3)}{e^{4x+4}} \\
\text{OR}
\end{align*}
\]

\[
\begin{align*}
f(g(x)) &= \ln(g(x) + 3) = \frac{\ln(e^{4x+4} + 3)}{e^{4x+4} + 3}
\end{align*}
\]

\[E: g, \quad f(x) = 2x+1, \quad g(x) = \ln(x+3), \quad h(x) = e^{4x+4} \]

\[
\begin{align*}
h(g(f(x))) &= h(\ln(f(x)+3)) \\
&= e^{4\ln(f(x)+3) + 7} \\
&= e^{4\ln(2x+4) + 7} \\
&= \ln((2x+4)^4 + 7)
\end{align*}
\]

\[
\begin{align*}
e^{\ln(x^4)} &= x^4 \\
e^{x+y} &= e^x e^y \\
e^{\ln(x)} &= x
\end{align*}
\]

\[E: g, \quad f(x) = x+1, \quad g(x) = x^2 + 1 \]

\[
\begin{align*}
g(f(x)) &= (f(x))^2 + 1 = (x+1)^2 + 1 \\
&= x^2 + 2x + 1 + 1 = x^2 + 2x + 2
\end{align*}
\]
6. **Absolute value of a function**: \( y = |f(x)| \)

Makes any negative values of \( f(x) \) positive and leaves positive values alone. "Brings" all of the graph of \( f(x) \) above the \( x \)-axis by reflecting all parts of the graph below the \( x \)-axis across the \( x \)-axis.

*Example:*

\[ y = \sin x \]

\[ y = |\sin x| \]
Recall: Linear Functions

They have a special property that I did not discuss, but that comes from the fact that the slope of a line is the same anywhere along the line.

E.g., \( f(x) = 3x + 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = f(x) )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>14</td>
</tr>
</tbody>
</table>

Successive differences are constant over equally spaced \( x \) values

Linear functions do not always model reality accurately. (One exception: Newton's Second Law of Motion \( F = ma \))

What models reality more accurately, but still not perfectly, are
EXponential Functions

Exponential functions are of the form

\[ f(x) = ka^x \]

- \( k \) is a constant
- \( a \) is a constant
- \( a > 0, a \neq 1 \)
- \( x \) is real

Q: Why is \( a > 0 \)?

A: • If \( a < 0 \), could have, e.g.,

\[ f(x) = 3(-4)^x \]

\[ f\left(\frac{1}{2}\right) = 3(-4)^{\frac{1}{2}} = 3 \sqrt{-4} = \text{undefined} \]

• If \( a = 0 \), have

\[ f(x) = k(0)^x = k \cdot 0 = 0 \]

which is not very interesting and we already have \( f(x) = 0 \) under the category of constant functions which, in turn, is in the category of polynomial functions (\( f(x) = 0 \cdot x^0 \)).
Q: Why is $a \neq 1$?

A: If $a = 1$, we have

$$f(x) = k (1)^x = k \cdot 1 = k,$$

which is also not very interesting and we already have $f(x) = 1$
under the category of constant functions which, in turn, is in
the category of polynomial functions ($f(x) = 1 \cdot x^0$).
**Exponential vs. Linear**

E.g., \( f(x) = 2^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>( y = 2^x )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

Successive ratios are constant over equally-spaced \( x \) values
Exponential vs. Power

The graph of any exponential function will eventually rise above the graph of any power function and stay that way.

Example: \( f(x) = 3^x \) eventually "dominates" \( f(x) = x^4 \):

![Graph showing exponential and power functions]
Exponential Functions Used in Modeling

E.g., Exponential growth of a population of wild rabbits (which doubles every week)

\[ P(t) = 2 (2)^t \]

E.g., Exponential decay of a radioactive substance (which decreases in amount by \( \frac{1}{2} \) every year)

\[ P(t) = 5 \left( \frac{1}{2} \right)^t = 5 (2^{-t}) = 5 (2)^{-t} \]
Standard Formulas for Half-Life and Doubling Time

**Exponential Growth:**

\[ P(t) = P_0 (2)^{t/H} \]

- \( H \) = doubling time
- (time it takes for the number or amount to double)
- \( P_0 = P(0) \)
- = initial number or amount present

**Exponential Decay:**

\[ P(t) = P_0 \left(\frac{1}{2}\right)^{t/h} \]

- \( h \) = half-life
- (time it takes for amount to be reduced by \( \frac{1}{2} \))
- \( P_0 = P(0) \)
- = initial amount present

**Examples:**

1. (HW Exercise 23, p. 62.)
HINT: Use the growth formula with $H = 3$ and $P_0 = 100$.

HINT: Use the decay formula with $h = 15$ and $P_0 = 2$. 

(HW Exercise 24, p. 62.)
First a definition.

Definition. A graph is **concave up** if any line segment connecting two points of the graph lies above the graph:

![Graph](image)

A graph is **concave down** if any line segment connecting two points of the graph lies below the graph:

![Graph](image)
Example. \( f(x) = 3^x \) \( (a = 3 > 1) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 3^x )</td>
<td>( 3^{-2} = \frac{1}{3^2} = \frac{1}{9} )</td>
<td>( 3^{-1} = \frac{1}{3} )</td>
<td>( 3^0 = 1 )</td>
<td>( 3^1 = 3 )</td>
<td>( 3^2 = 9 )</td>
</tr>
</tbody>
</table>

**Features of graph**
- \( y \)-int. = 1
- Horiz. Asympt.:
  - \( y = 0 \) (x-axis)
- Increasing (rapidly)
- Concave up

Example. \( f(x) = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x} \) \( (a = \frac{1}{3} < 1) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 3^{-x} )</td>
<td>( 3^{-(-1)} = 3^1 = 3 )</td>
<td>( 3^{(-0)} = 3^0 = 1 )</td>
<td>( 3^{-1} = \frac{1}{3} )</td>
</tr>
</tbody>
</table>

**Features of graph**
- \( y \)-int. = 1
- Horiz. Asympt.:
  - \( y = 0 \) (x-axis)
- Decreasing (rapidly)
- Concave up
Example. Graph $y = 3 - 2e^{x+1}$ without using a calculator.

\[ y = 3 - 2e^{x+1} \Rightarrow y - 3 = -2e^{x+1} \]

1. Shift vertically up 3 units (do first).
2. Reflect vertically across x-axis.
3. Reflect across x-axis.
4. Stretch vertically.
5. Shift horizontally to the left 1 unit.
The equation is:

\[ y = 3 - 2e^{x+1} \]
Section 1.6. Inverse Functions and Logarithms.

Two functions are "inverses" of each other if either one "undoes" what the other is doing to $x$.

E.g., $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

$$f(g(x)) = f(\sqrt[3]{x^3}) = (\sqrt[3]{x^3})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

**Notation:** Inverse of $f(x)$ is $f^{-1}(x)$, and vice versa.

**Warning:** $f^{-1}(x) \neq \frac{1}{f(x)} = \text{reciprocal of } f(x)$

(But $(f(x))^{-1} = \frac{1}{f(x)}$)

**Terminology:** $f(x)$ is said to be invertible if it has an inverse.
3 Ways to Represent Inverses of Functions

1. Algebraically:

\( f(x) \) and \( f^{-1}(x) \) are inverses if

1. \( f(f^{-1}(x)) = x \)
2. \( f^{-1}(f(x)) = x \)

E.g., \( f(x) = x + 1 \) \( g(x) = x - 1 \)

\( f(g(x)) = f(x - 1) = (x - 1) + 1 = x \checkmark \)

\( g(f(x)) = g(x + 1) = (x + 1) - 1 = x \checkmark \)

\( \therefore \ g(x) = f^{-1}(x) \)

2. In Tabular Form:

E.g.,

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
</tr>
</tbody>
</table>

\[ \text{Domain} \rightarrow \text{Range} \rightarrow \text{Domain} \]

Switch Domain and Range of \( f \) to get \( f^{-1} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f^{-1}(x) = \sqrt{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>3</td>
</tr>
<tr>
<td>64</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \text{Domain} \rightarrow \text{Range} \rightarrow \text{Domain} \]

Domain \( f = \text{Range} \) \( f^{-1} \)

Range \( f = \text{Domain} \) \( f^{-1} \)
3. **GRAPHICALLY:**

\(f(x)\) and \(f^{-1}(x)\) are inverses if the following is true:

If \((a, b)\) is a point on the graph of \(y = f(x)\) (so \(b = f(a)\)), then \((b, a)\) is a point on the graph of \(y = f^{-1}(x)\) (so \(a = f(b)\)).

This can only happen if the graph of \(y = f^{-1}(x)\) is a reflection of the graph of \(y = f(x)\) about the diagonal line \(y = x\).

\[\text{E.g.}\]

\[
\begin{align*}
&\text{Graph of } y = x^5 \\
&\text{Graph of } y = x \\
&(2, 32) \\
&(2, 5)
\end{align*}
\]
How to Derive $f^{-1}(x)$ From $f(x)$

E.g., $f(x) = \frac{1}{x+1}$

**STEP 1.** Replace $f(x)$ by $y$:

$$y = \frac{1}{x+1}$$

**STEP 2.** Switch $x$'s and $y$'s:

$$x = \frac{1}{y+1}$$

**STEP 3.** Solve for $y$:

$$x = \frac{1}{y+1} \Rightarrow x(y+1) = 1 \Rightarrow xy + x = 1 \Rightarrow xy = 1 - x \Rightarrow y = \frac{1-x}{x} \quad (= \frac{1}{x} - 1)$$

**STEP 4.** Replace $y$ by $f^{-1}(x)$:

$$f^{-1}(x) = \frac{1-x}{x}$$
Graphical Test for Invertibility

Test to see if graph belongs to a function:

VERTICAL LINE TEST

E.g., \( y = e^x \)

- Of a function
  - Only one \( y \) for each \( x \)

- Not of a function
  - As many as 2 \( y \)'s for each \( x \)

Test to see if graph belongs to an invertible function:

HORIZONTAL LINE TEST

- Of an invertible function
  - Only one \( x \) for each \( y \)

- Not of an invertible function
  - As many as 2 \( x \)'s for each \( y \)
The Logarithmic Function

The logarithmic function is the inverse of the exponential function. Let us try to derive a formula for the inverse of \( f(x) = a^x \):

**STEP 1.** Replace \( f(x) \) by \( y \):

\[ y = a^x \]

**STEP 2.** Switch \( x \)'s and \( y \)'s:

\[ x = a^y \]

**STEP 3.** Solve for \( y \):

\[ x = a^y \Rightarrow \text{CANNOT solve for } y \]

using operations of addition/subtraction and multiplication/division.

\[ \times \]

**INSTEAD:** We stop at **STEP 2** and do not solve for \( y \).
Definition. The logarithm to base $a$ of $x$, written $\log_a x$, is the power to which $a$ must be raised to get $x$, i.e.,

$$\log_a x = y \quad \text{means} \quad a^y = x$$

So, if $f(x) = a^x$, we say that $f^{-1}(x) = \log_a x$.

Warning: $y = \log_a x$ is ONLY defined for $x > 0$!

$$\log_a 0 \quad \log_a (-1)$$

Example. Compute $\log_3 27$.

1st Set $\log_3 27 = y$

2nd $\log_3 27 = y$ means $3^y = 27$
3rd Guess that \( y \) has to be 3
\[ \log_3 {27} = 3 \]
Two Important Log Functions
(On Your Calculators)

\[ \log x = \text{COMMON LOG} \]
\[ \text{to the base } 10 \]
\[ \log = \log_{10} \]
\[ \text{inverse of } \log_{10} x \text{ is } 10^x \]

\[ \ln x = \text{NATURAL LOG} \]
\[ \text{to the base } e \]
\[ \ln = \log_{e} \]
\[ \text{inverse of } \log_{e} x \text{ is } e^x \]

Formula to convert \( \log_a x \) to \( \ln \):

\[ \log_a x = \frac{\ln x}{\ln a} \]

(You can plug this into calculator)
### Some Important Rules of Logs and Exponents

<table>
<thead>
<tr>
<th>Exponents</th>
<th>Logs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $a^x a^y = a^{x+y}$</td>
<td>1. $\log_a x y = \log_a x + \log_a y$</td>
</tr>
<tr>
<td>2. $(a^x)^r = a^{xr}$</td>
<td>2. $r \log_a x = \log_a x^r$</td>
</tr>
<tr>
<td>3. $a^\log_a x = x$</td>
<td>3. $\log_a a^x = x$</td>
</tr>
<tr>
<td>4. $a^0 = 1$</td>
<td>4. $\log_a 1 = 0$</td>
</tr>
</tbody>
</table>

- 2/2/01 Lecture
Graphs of Log Functions

\[ f(x) = \log_a x, \quad a > 1 \]

Eq. \( f(x) = \log x \), \( f(x) = \ln x \)

Features:
- Domain = \( \{ x | x > 0 \} \)
- Range = all reals
- \( x \)-intercept = 1
- Vertical asymptote: \( y \)-axis \( (x = 0) \)
- Increasing (but very gradually)
- Concave down
Will OMIT graph of \( f(x) = \log_a x \) when \( 0 < a < 1 \).
CHAPTER 2. Limits and Derivatives.

We will first define

**LIMIT OF A FUNCTION.**

From this we will give a precise definition of

**CONTINUOUS FUNCTION**

and

**DIFFERENTIABLE FUNCTION**

(or Function with a Derivative)

Continuous functions have nice features, but differentiable functions have even nicer features.

Calculus is divided into two main areas:

Functions with Derivatives and Functions with Integrals.
Section 2.2. The limit of a Function.

Example. Suppose you are asked to evaluate the (rational) function

\[ f(x) = \frac{x+2}{x^2-4} \]

at the point \( x = -2 \) (i.e., \( f(-2) = ? \)).

If you directly plug \( x = -2 \) into

\[ f(x) = \frac{x+2}{x^2-4} \]

you get something nonsensical:

\[ f(-2) = \frac{(-2)+2}{(-2)^2-4} = \frac{0}{0} \]

(this makes even less sense than something like \( \frac{1}{0} \), which we sometimes say is "equal to infinity").

However, there is a way around this which gives the next best thing to
to saying what \( f(x) \) is at \( x = -2 \);

Instead of saying what \( f(x) \) IS WHEN \( x = -2 \), we say what \( f(x) \) APPROACHES AS \( x \) APPROACHES (but NEVER QUITE REACHES) THE VALUE \(-2\).

So, we do the following:

- Make a TABLE of \( x \) versus \( f(x) \)
- GUESS at what \( f(x) \) is approaching as \( x \) approaches (but never reaches) \(-2\)

```
-3 -2 -1 0
```

![Graph showing x-values from -3 to 0 with arrows indicating direction of approach to -2.](image-url)
\[
x = \begin{array}{c|c}
-1 & 0.3333333 \\
-1.9 & 0.25641 \\
-1.99 & 0.250627 \\
-1.999 & 0.250063 \\
-1.9999 & 0.250006 \\
-2 & 0.25 \ (\text{guess})
\end{array}
\]

\[
x = \begin{array}{c|c}
-3 & -0.2 \\
-2.1 & -0.243902 \\
-2.01 & -0.249377 \\
-2.001 & -0.249938 \\
-2.0001 & -0.249994 \\
-2 & 0.25 \ (\text{guess})
\end{array}
\]

We then write

\[f(x) \to -0.25 \quad \text{as} \quad x \to -2\]

"\(f(x)\) approaches the value \(-0.25\)" \hspace{1cm} "\(x\) approaches the value \(-2\)"
or

\[
\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x+2}{x^2-4} = -0.25
\]

"The limit of \( f(x) \) or \( \frac{x+2}{x^2-4} \) as \( x \) approaches (but never reaches) -2 is equal to -0.25"

We never consider \( x = -2 \), since \( f(x) \) may (or may not) be defined at \( x = -2 \), but it is defined everywhere around \( x = -2 \).

Actually, in many cases we can avoid the messy TABLE and the GUESSING and instead do the following:

- Use ALGEBRA
- Evaluate the limit of \( f(x) \) EXACTLY
\[ \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x + 2}{x^2 - 4} \]

Do algebra "inside" the limit, where \( x \) takes on values getting closer to 2 but never equal to 2.

\[ \lim_{x \to 2} \frac{x + 2}{(x - 2)(x + 2)} \]

Can cancel the \( x + 2 \) since \( x \) is never equal to -2 \( \Rightarrow \) \( x \neq -2 \)

\( \Rightarrow \) \( x + 2 \neq 0 \) and anything except 0 can be cancelled.

\[ \lim_{x \to 2} \frac{1}{x - 2} \]

As long the resulting expression no longer equals \( \frac{0}{0} \) when \( x = -2 \), then one can say that the limit of \( \frac{1}{x - 2} \) as \( x \) approaches (but never reaches) -2 is equal to \( \frac{-1}{-4} \) when \( x = -2 \).

\[ \frac{1}{-2 - 2} = \frac{1}{-4} = \frac{-0.25}{-0.25} \]
\[
\lim_{{x \to -2^-}} f(x) = \text{LEFT-HAND LIMIT OF } f(x)
\]

\[
\lim_{{x \to -2^+}} f(x) = \text{RIGHT-HAND LIMIT OF } f(x)
\]

\[
\text{x approaches -2 from values less than or to the left of -2}
\]

\[
\text{x approaches -2 from values greater than or to the right of -2}
\]

\[
-0.25 \quad (\text{SEE previous TABLE})
\]

A function \( f(x) \) does not necessarily have a limit as \( x \) approaches some value \( a \).
Definition. \( \lim_{x \to a} f(x) \) is said to exist if \( f(x) \) approaches a single finite value as \( x \) approaches \( a \).

Examples.

1. \( f(x) = \frac{\sin x}{x} \) (\( x \) in radians!)

\[
\lim_{x \to 0} \frac{\sin x}{x} = ?
\]

Note that \( f(0) = \frac{\sin 0}{0} = \frac{0}{0} = ? \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \frac{\sin x}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.841471</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.9783342</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.9999833</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.99999798</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
\[ f(x) = \frac{\sin x}{x} \]

<table>
<thead>
<tr>
<th>0.001</th>
<th>0.9997998</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.9999833</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9988342</td>
</tr>
<tr>
<td>1</td>
<td>0.841471</td>
</tr>
</tbody>
</table>

\[
\lim_\limits{x \to 0} \frac{\sin x}{x} = 1
\]

We can say with some confidence that

A single, finite number, so this limit exists.
2. \[ f(x) = \frac{1}{x^2} \]

\[ \lim_{x \to 0} \frac{1}{x^2} = ? \]

\[ \frac{1}{x} \text{ gets larger and larger as } x \text{ gets smaller and smaller} \]
We can say with some confidence that

\[ \lim_{x \to 0} \frac{1}{x^2} = +\infty \]

This is a single but not finite value. So this limit does not exist (DNE).
(We can also say: This limit is infinite.)

Note that \( \frac{1}{0^2} = \frac{1}{0} = \infty \) (actually, +\( \infty \))
3. \( f(x) = \begin{cases} 
  x^2, & \text{if } x < 0 \\
  x + 1, & \text{if } x \geq 0 
\end{cases} \)  

\text{MULTIPLY-DEFINED FUNCTION}

\[
\begin{align*}
  \text{if } x < 0: & \quad y = x^2 \\
  \text{if } x \geq 0: & \quad y = x + 1
\end{align*}
\]

\( f(x) \to 0 \text{ as } x \to 0^- \)  
(\text{but } f(0) \text{ never reaches 0})

\( f(x) \to 1 \text{ as } x \to 0^+ \)  
(\text{and } f(x) \text{ reaches 1 when } x = 0)

\[ \lim_{x \to 0} f(x) = ? \]

From the graph, we see that

\[
\begin{align*}
  \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} x^2 = (0)^2 = 0 \\
  \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x+1) = (0)+1 = 1
\end{align*}
\]

\[
\therefore \quad \lim_{x \to 0} f(x) = \begin{cases} 
  0 & \text{or} \\
  1 & \text{or} \end{cases}
\]

This is not a \underline{single value} (although both are \underline{finite}).
So this limit \underline{does not exist} (DNE).
Theorem. If
\[ \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x), \]
then
\[ \lim_{x \to a} f(x) \text{ DNE.} \]

If
\[ \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L, \]
then
\[ \lim_{x \to a} f(x) = L \text{ and exists.} \]

Notes,
1. If \( f(a) = \frac{0}{0} \), there is a chance that \( \lim_{x \to a} f(x) \) exists.
E.g., \( f(x) = \frac{x+1}{x^2+3x+2} \)

\[ f(-1) = \frac{0}{0} \]

But \( \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x+1}{x^2+3x+2} \)

\[ = \lim_{x \to -1} \frac{x+1}{(x+1)(x+2)} \]

\[ = \lim_{x \to -1} \frac{1}{x+2} \]

\[ = \frac{1}{(-1)+2} \]

\[ = \frac{1}{1} \]

\[ = 1 \]

2. \( \frac{0}{0} = \text{non sense} \)

\( \frac{1}{0} = \infty \text{ or undefined} \)

\( \frac{0}{1} = 0 \text{ and defined} \)
Examples.

1. (SEE HW Exercise 1, p. 109.)

Explain in words what is meant by the equation

\[ \lim_{x \to -2} f(x) = 6. \]

Is it possible for this statement to be true and yet \( f(-2) = 1 \)? Explain.

"\( \lim_{x \to -2} f(x) = 6 \)" means that

The value of \( f(x) \) approaches the value of 6 as \( x \) approaches (but never equals) the value of \(-2\) (from either the right or the left).

"\( f(-2) = 1 \)" means that

The value of \( f(x) \) is 1 when the value of \( x \) is \(-2\).

It is possible for "\( \lim_{x \to -2} f(x) = 6 \)" to be true and yet "\( f(-2) = 1 \)," e.g.,
\[ f(x) = \begin{cases} 
  x + 8, & \text{if } x \neq -2 \\
  1, & \text{if } x = -2 
\end{cases} \]

When \( x = -2 \) but \( x \) approaches \(-2\), \( f(x) \) approaches what looks like \( 6 \).

\[ f(-2) \text{ is defined to be } 1 \]

\[
\begin{pmatrix}
  x \\
  y = x + 8 \\
  0 \\
  8 \\
  -2 \\
  6
\end{pmatrix}
\]
2. (SEE HW Exercise 2, p. 109)

Explain what it means to say that
\[ \lim_{x \to 10^-} f(x) = 100 \quad \text{and} \quad \lim_{x \to 10^+} f(x) = 50. \]

In this situation, is it possible that \( \lim_{x \to 10} f(x) \) exists? Explain.

\[ \lim_{x \to 10^-} f(x) = 100 \quad \lim_{x \to 10^+} f(x) = 50 \]

Value of \( f(x) \) approaches 100 as \( x \) approaches 10 from the left (or from values less than 10).

Value of \( f(x) \) approaches 50 as \( x \) approaches 10 from the right.

It is not possible for \( \lim_{x \to 10} f(x) \) to exist since

\[ \lim_{x \to 10^-} f(x) \neq \lim_{x \to 10^+} f(x) \]

\[ \frac{100}{100} \neq \frac{50}{50} \]
\[ E \{ q \} f(x) = \begin{cases} 
9x + 10 & \text{if } x \leq 10 \\
50 & \text{if } x \geq 10 
\end{cases} \]
State the value of the limit, if it exists, from the given graph. If it does not exist, explain why.

(a) \( \lim_{x \to 3} f(x) \)  
(b) \( \lim_{x \to 1} f(x) \)  
(c) \( \lim_{x \to -3} f(x) \)  
(d) \( \lim_{x \to -2} f(x) \)  
(e) \( \lim_{x \to 2^+} f(x) \)  
(f) \( \lim_{x \to 2^-} f(x) \)  
(g) \( \lim_{x \to 2} f(x) \)
(a) \( \lim_{x \to 3} f(x) = 2 \) since
\[
\lim_{x \to 3^-} f(x) = 2 = \lim_{x \to 3^+} f(x)
\]

(In fact, \( f(3) = 2 \) in this particular case!)

(b) \( \lim_{x \to 1} f(x) = -1 \) (but \( f(1) = 1 \))
(c) \( \lim_{{x \to -3}} f(x) = 1 \) (but \( f(x) \) is undefined at \( x = -3 \))

since

\[
\lim_{{x \to -3^-}} f(x) = 1 = \lim_{{x \to -3^+}} f(x)
\]

\( f(x) \) not defined here

(d) \( \lim_{{x \to 2^-}} f(x) = 1 \) since

\[
\lim_{{x \to 2^-}} f(x) = 1
\]

\( f(x) \)
(e) \( \lim_{x \to 2^+} f(x) = 2 \) since

(\text{graph showing limit from the right)}

(f) \( \lim_{x \to 2} f(x) \) DOES NOT EXIST (DNE) since

\[
\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) = 2
\]
Use a graph to determine how close to 0 we have to take \( x \) to ensure that \( e^x \) is within a distance 0.2 of the number 1.

What if we insist that \( e^x \) be within 0.2 of 1?

When \( a < x < b \), then \( 0.8 < e^x < 1.2 \).

We need to find or guess at \( a \) and \( b \).
So, we graph three functions with our graphing calculator:

\[ y_1 = e^x \]
\[ y_2 = 0.8 \quad \text{(horizontal line at 0.8)} \]
\[ y_3 = 1.2 \quad \text{(horizontal line at 1.2)} \]

Try window dimensions of

\[ x_{\text{Min}} = -2 \]
\[ x_{\text{Max}} = 2 \]
\[ y_{\text{Min}} = 0 \quad (\leq 0.8) \]
\[ y_{\text{Max}} = 2 \quad (> 1.2) \]

Then get

\[ y = e^x \]
\[ y = 1.2 \]
\[ y = 0.8 \]

Use TRACE. Step when
\[ y = 0.8 \]

Use TRACE. Step when
\[ y = 1.2 \]

\[ x = -0.212766 \]
\[ y = 0.8 \]

\[ x = 0.17021277 \]
\[ y = 1.2 \]
To have

\[ 0.8 < e^x < 1.2, \]

we need (approximately)

\[ -0.21 < x < 0.17 \Rightarrow \]

\[ 0 - 0.21 < x < 0 + 0.17 \]

If we require \( x \) to be within 0.17 of 0 (choose the smaller of 0.17 and 0.21 to play it safe), we ensure that \( y = e^x \) is within 1.2 of 1, i.e.,

\[ \frac{0.17}{-0.17} \]

will guarantee that

\[ \frac{0.2}{0.8} \]
Section 2.3: Calculating Limits Using Limit Laws.

Limit laws provide us with the following:

1. Algebraic rules for manipulating limits;
2. The values of some special limits.

Some Limit Laws

1. \( \lim_{x \to a} c = c \)

E.g., \( \lim_{x \to 0} 2 = 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -0.1 )</td>
<td>2</td>
</tr>
<tr>
<td>( -0.01 )</td>
<td>2</td>
</tr>
<tr>
<td>( -0.001 )</td>
<td>2</td>
</tr>
<tr>
<td>( 0.1 )</td>
<td>2</td>
</tr>
<tr>
<td>( 0.01 )</td>
<td>2</td>
</tr>
<tr>
<td>( 0.001 )</td>
<td>2</td>
</tr>
</tbody>
</table>

Graphically:

\[ f(x) \to 2 \quad \text{as} \quad x \to a^+ \]

<table>
<thead>
<tr>
<th>( x \to a^+ )</th>
<th>( y = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Tables:
2. \( \lim_{x \to a} x = a \)

Example: \( \lim_{x \to 3} x = 3 \)

Graphically:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x )</th>
<th>( f(x) = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9</td>
<td>2.9</td>
<td>3.0</td>
</tr>
<tr>
<td>2.99</td>
<td>2.99</td>
<td>3.01</td>
</tr>
<tr>
<td>2.999</td>
<td>2.999</td>
<td>3.001</td>
</tr>
<tr>
<td>3^-</td>
<td>3</td>
<td>3^+</td>
</tr>
</tbody>
</table>

3. \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \)

Example:

\[
\lim_{x \to 2} \left[ \sqrt{x-1} + \sqrt{x+1} \right] = \lim_{x \to 2} \sqrt{x-1} + \lim_{x \to 2} \sqrt{x+1} = \sqrt{2-1} + \sqrt{2+1} = 1 + \sqrt{3}
\]
4. \( \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) \)

E.g., \( \lim_{x \to 4} \frac{1}{2} x = \frac{1}{2} \lim_{x \to 4} x = \frac{1}{2} (4) = 2 \)

5. \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \)

E.g., \( \lim_{x \to -2} x^2 = \lim_{x \to -2} (x \cdot x) \)
\( \quad = \lim_{x \to -2} x \cdot \lim_{x \to -2} x \)
\( \quad = (-2) (-2) \)
\( \quad = 4 \)

6. \( \lim_{x \to a} x^n = a^n \)

E.g., \( \lim_{x \to 2} x^5 \)
\( \quad = \lim_{x \to 2} x \cdot \lim_{x \to 2} x \cdot \lim_{x \to 2} x \cdot \lim_{x \to 2} x \cdot \lim_{x \to 2} x \)
\( \quad = (\lim_{x \to 2} x)^5 \)
\( \quad = (2)^5 \)
\( \quad = 32 \)
\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0 \]

E.g., \[ \lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} \]

\[ \lim_{x \to 1} (x^2-1) = (1)^2-1 = 1-1 = 0 \]

No, since \( \lim_{x \to 1} (x^2-1) = 0 \)

So, start again:

\[ \lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} \]

Yes, since \( \lim_{x \to 1} (x+1) = 2 \neq 0 \)

\[ 1+1 = 2 \neq 0 \]
\[ \lim_{x \to a} f(x) = \lim_{x \to a} f(x) \]

E.g., \[ \lim_{x \to 5^+} \sqrt{x^2 - 25} = \lim_{x \to 5^+} \sqrt{(x^2 - 25)} \]
\[
\begin{align*}
5 & \quad \text{if } x > 5 \\
5^+ & \quad \text{if } x > 5^+ \\
x^2 > 25 & \Rightarrow \text{ makes sense} \\
x^2 - 25 > 0 & \Rightarrow \text{ sense} \\
\lim_{x \to 5^+} \sqrt{x^2 - 25} & = 0
\end{align*}
\]
Examples.

1. (HW Exercise 3, p. 118.)
\[
\lim_{x \to 4} (5x^2 - 2x + 3) = ?
\]

\[
\lim_{x \to 4} (5x^2 - 2x + 3) = 5(4)^2 - 2(4) + 3
\]
\[
= 80 - 8 + 3
\]
\[
= 75
\]

Or

\[
\lim_{x \to 4} (5x^2 - 2x + 3) = 5 \left( \lim_{x \to 4} x \right)^2 - 2 \cdot \lim_{x \to 4} x
\]
\[
+ \lim_{x \to 4} 3
\]
\[
= 5(4)^2 - 2(4) + 3
\]
\[
= 80 - 8 + 3
\]
\[
= 75
\]

2. (HW Exercise 11, p. 118.)
\[
\lim_{h \to 0} \frac{(h-5)^2 - 25}{h}
\]
\[ \lim_{h \to 0} \frac{(h-5)^2 - 25}{h} = \lim_{h \to 0} \frac{h^2 - 10h + 25 - 25}{h} = \lim_{h \to 0} \frac{h^2 - 10h}{h} \]

\[ = \lim_{h \to 0} (h - 10) \]

\[ = 0 - 10 \]

\[ = -10 \]

\[ \text{NOTE: What we really have is} \]

\[ \lim_{h \to 0} \frac{f(-5 + h) - f(-5)}{h} \]

where

\[ f(x) = x^2, \quad f(-5) = (-5)^2 = 25, \]

\[ f(-5 + h) = (-5 + h)^2 = (h-5)^2 \]

This quotient is called

THE DERIVATIVE OF \( f(x) \)

AT \( x = -5 \)

and is denoted by

\[ f'(-5) \]
3. (HW Exercise 13, p. 118)

\[ \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} = ? \]

\[ \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} = \lim_{t \to 9} \frac{(3 - \sqrt{t})(3 + \sqrt{t})}{3 - \sqrt{t}} \]

\[ \frac{9 - t}{3 - \sqrt{t}} = \frac{0}{0} \]

\[ \lim_{t \to 9} (3 + \sqrt{t}) = 3 + 3 = \frac{6}{1} \]

4. (HW Exercise 14, p. 118)

\[ \lim_{x \to 1} \left( \frac{1}{x-1} - \frac{2}{x^2-1} \right) = ? \]

\[ \lim_{x \to 1} \left( \frac{1}{x-1} - \frac{2}{x^2-1} \right) = \lim_{x \to 1} \left[ \frac{1}{x-1} - \frac{2}{(x-1)(x+1)} \right] \]

Make one fraction out of these
\[
\lim_{x \to 1} \left[ \frac{1}{x-1} \frac{x+1}{x+1} - \frac{2}{(x-1)(x+1)} \right]
\]
\[
= \lim_{x \to 1} \frac{x+1-2}{(x-1)(x+1)}
\]
\[
= \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)}
\]
\[
= \lim_{x \to 1} \frac{1}{x+1}
\]
\[
= \frac{1}{1+1} = \frac{1}{2}
\]
The Squeeze Theorem

Let \( f(x) \), \( g(x) \), \( h(x) \) be 3 functions where
\[
f(x) \leq g(x) \leq h(x).
\]
Then
\[
(1) \quad \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x).
\]
\[
(2) \quad \text{if } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L
\]
then \( \lim_{x \to a} g(x) = L \).

E.g., \( f(x) = 2x^3 \), \( g(x) = x^3 \), \( h(x) = \frac{1}{2}x^3 \)
\[ f(x) \leq g(x) \leq h(x) \quad \text{for all } x \geq 0 \Rightarrow \]
\[ \lim_{x \to 0^+} f(x) \leq \lim_{x \to 0^+} g(x) \leq \lim_{x \to 0^+} h(x) \Rightarrow \]
\[ 0 \leq 0 \leq 0 \]
\[ \lim_{x \to 0^+} g(x) = 0 \text{ by the Squeeze Theorem} \]

Note. The Squeeze Theorem is useful when you have a function, \( g(x) \), whose limit is difficult to find, but which can be compared to other functions whose limits are easy to find.

Example. (HW Exercise 19, p. 119,)
Use the Squeeze Theorem to show
\[ \lim_{x \to 0} x^2 \cos(20\pi x) = 0. \]

Let \( g(x) = x^2 \cos(20\pi x) \). Find an \( f(x) \) and \( h(x) \) such that
\[ f(x) \leq g(x) \leq h(x), \]

---
Note that

\[-1 \leq \cos(bx), \sin(bx) \leq 1 \Rightarrow\]

\[-1 \leq \cos(20\pi x) \leq 1 \Rightarrow\]

\[-1 \times x^2 \leq x^2 \cos(20\pi x) \leq (1) \times x^2 \Rightarrow\]

\[-x^2 \leq \cos(20\pi x) \leq x^2 \]

Let \( f(x) = -x^2 \) \quad \text{and} \quad \text{Let} \ k(x) = x^2 \]

So

\[-x^2 \leq \cos(20\pi x) \leq x^2 \Rightarrow\]

\[
\lim_{x \to 0} (-x^2) \leq \lim_{x \to 0} x^2 \cos(20\pi x) \leq \lim_{x \to 0} x^2 \Rightarrow
\]

\[
-(0)^2 \leq 0 \leq (0)^2
\]

\[
\lim_{x \to 0} x^2 \cos(20\pi x) = 0
\]
Section 2.4, Continuity.

A function is considered to be continuous (on some interval) if its graph is connected (over that interval), i.e., has no breaks, holes, or jumps.

E.g., A graph with breaks:

![Graph with breaks](image)

E.g., A graph with holes:

![Graph with holes](image)
E.g., A graph with jumps:

![Graph with jumps](image)

---

**Formal Definition of Continuity of a Function**

**Defn.** A function \( f \) is continuous at a point \( x = a \) if

\[
\lim_{{x \to a}} f(x) = f(a).
\]
This can be broken down into a "working" definition, whose steps can be easily checked:

**Defn.** \( f \) is continuous at a point \( x = a \) if

1. \( f(a) \) is defined (i.e., \( f(a) \) is assigned a finite number and not \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \))
2. \( \lim_{x \to a} f(x) \) exists (i.e., \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \))
3. \( \lim_{x \to a} f(x) = f(a) \)

**Note.** A function can be "half continuous at a point:

**Defn.** \( f \) is continuous from the right at \( x = a \) if

\[ \lim_{x \to a^+} f(x) = f(a) \]

\( f \) is continuous from the left at \( x = a \) if

\[ \lim_{x \to a^-} f(x) = f(a) \]
E.g., \( f(a) \) is undefined, but \( \lim_{x \to a} f(x) \) exists:

\[
f(x) = \frac{x^2 - 1}{x + 1} \quad a = -1
\]

1. \( f(-1) = \frac{(-1)^2 - 1}{(-1) + 1} = \frac{0}{0} \)
   
   \( f(-1) \) is undefined

2. \( \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x + 1)(x - 1)}{x + 1} \)
   
   \( \lim_{x \to -1} (x - 1) = (-1) - 1 = -2 \)

   \( \therefore \lim_{x \to -1} f(x) \) exists

E.g., \( f(a) \) is defined, but \( \lim_{x \to a} f(x) \) does not exist:

\[
f(x) = \begin{cases} 
-5 & \text{if } x < 0, \\
5 & \text{if } x \geq 0 
\end{cases} \quad a = 0
\]

1. \( f(0) = 5 \) (where \( f(x) = 5 \) for all \( x \geq 0 \))
   
   \( f(0) \) is defined

2. \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -5 = -5 \)
   
   \( \therefore f(x) = -5 \)
\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 5 = 5 \]

\[ \Rightarrow x \geq 0 \]

\[ \Rightarrow f(x) = 5 \]

Since \( \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \),
\( \lim_{x \to 0} f(x) \) does not exist

\[ \text{E.g., } f(x) \text{ is defined, } \lim_{x \to a} f(x) \text{ exists, but } \lim_{x \to a} f(x) \neq f(a); \]

\[ f(x) = \begin{cases} 
\frac{x^2 - 1}{x + 1}, & \text{if } x \neq -1 \\
3, & \text{if } x = -1 
\end{cases} \quad a = -1 \]

1. \( f(-1) = 3 \)
   \( \therefore f(-1) \) is defined
2. \[ \lim_{{x \to -1}} f(x) = \lim_{{x \to -1}} \frac{x^2 - 1}{x + 1} = -2 \]

\[ \Rightarrow x \neq -1 \]

\[ \Rightarrow f(x) = \frac{x^2 - 1}{x + 1} \]

\[ \therefore \lim_{{x \to -1}} f(x) \text{ exists.} \]

\[ f(-1) = 3 \]

\[ f \text{ is not even from the right or left since } \]

\[ \lim_{{x \to -1}} f(x) = -2 \neq -3 = f(-1) \]

\[ \lim_{{x \to -1}} f(x) = -2 \neq 3 = f(-1) \]

\[ \lim_{{x \to -1}} f(x) = -2 \neq 3 = f(-1) \]
Shortcuts in Identifying Continuous Functions or Where Functions are Continuous

**Definition:** An interval is a connected portion of the x-axis.

- **Example:** $[-1, 1]$
- **Example:** $(3, 5)$
- **Example:** $(-7, 1]$
- **Example:** $[1, \infty)$
- **Example:** $(-\infty, 10)$
- **Example:** $(-\infty, \infty)$
Defn. A function $f$ is **continuous on** an interval $I$ if it is continuous at every point in $I$.

Note. If, for example, $f$ is continuous on $[a, b]$, then

1. For every point $x = c$ with $a < c < b$,
   \[
   \lim_{x \to c} f(x) = f(c)
   \]

2. For $x = a$,
   \[
   \lim_{x \to a^+} f(x) = f(a)
   \]

3. For $x = b$,
   \[
   \lim_{x \to b^-} f(x) = f(b)
   \]

Defn. A function $f$ is **continuous** if it is continuous on $(-\infty, \infty)$ (i.e., at all $x$).

Whole Classes of Functions Continuous on $(-\infty, \infty)$

- **Linear Functions**: 
  \[ f(x) = ax + b \]
  \[ \text{E.g., } f(x) = 3x + 1 \]

- **Polynomial Functions**: 
  \[ f(x) = ax^n + \ldots + b \]
  \[ \text{E.g., } f(x) = x^2 - 7x - 6 \]
Exponential Functions:

\[ f(x) = e^x - 5 \]

Sine/Cosine Functions:

\[ f(x) = 1 + 3 \sin(2x) \]

So, if you are able to identify a function \( f \) as belonging to one of the above classes, you can say it is continuous (for all \( x \)) without checking through the definition of continuity.

---

Rational Functions are Continuous at Every Point in Their Domain

Example. \( f(x) = \frac{1}{x-1} \)

\[ y \]
\[ x \]

\[ f(x) = \frac{1}{x-1} \]
Domain \( f = \{ x \mid x \neq 1 \} \)

\( f \) is continuous on \((-\infty, 1)\) and on \((1, \infty)\) (i.e., everywhere except at \( x = 1 \))

\( f \) is continuous on \((-\infty, 1) \cup (1, \infty)\) by union

---

Other Functions are Sometimes Continuous on Their Domains

Example. \( f(x) = \tan x \)

\[ \begin{align*}
\text{Domain } f &= \{ x \mid x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \} \\
\end{align*} \]
$f$ is continuous on

$\cdots (-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2}) \cdots$

But, for example, $f$ is not continuous from the left at $x = \frac{\pi}{2}$ since

1. $f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = -\infty$ is undefined

2. $\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$ does not exist

so cannot have

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$
Intermediate Value Theorem (IVT)

Theorem. If the graph of a continuous function has one point below and one point above the x-axis, then it must cross the x-axis.

\[ y \]
\[ 0 \quad x \]

or

\[ y \]
\[ 0 \quad x \]

Note. One can make use of this property to show that an equation has a solution.

Example. (Exercise 33, p. 129.)

Use the IVT to show that there is a root (= solution) of the equation \( x^3 - 3x + 1 = 0 \) on \((0, 1)\)

i.e., show there is a solution \(x\) between 0 and 1.
Set \( f(x) = x^2 - 3x + 1 \).

Since \( f \) is a polynomial function, it is continuous everywhere.

Notice: \( f(0) = (0)^2 - 3(0) + 1 = 1 > 0 \implies \) The graph of \( f \) has the point \((0,1)\), which is above the \( x \)-axis.

\[
f(1) = (1)^2 - 3(1) + 1 = -1 < 0 \implies \text{The graph of } f \text{ has the point } (1, -1), \text{ which is below the } x \text{-axis.}
\]

Since \( f \) is continuous, by the IVT, there exists a point \( x = c \) in the interval \((0, 1)\) where

\[
f(c) = 0
\]

(We can only guess at what \( c \) is.)
\[ f(c) = 0 \implies c^2 - 3c - 1 = 0 \]

The equation \( x^3 - 3x + 1 = 0 \) has a root \( x = c \) in the interval (0, 1).
Lecture

Section 2.5. Limits Involving Infinity.

"Limits involving infinity" fall into one of 3 categories:

**CATEGORY 1** \( \lim_{x \to a} f(x) = +\infty \) (or \( -\infty \))

"As \( x \) approaches the value \( a \), \( f(x) \) gets larger and larger (+\( \infty \)) (or more and more negative (-\( \infty \)))."

This limit really does not exist, but the above notation shows how it does not exist.

Use **TABLES** or **GRAPHS** to evaluate this kind of limit. I will use graphs.

**Examples.**

1. \( \lim_{x \to 0} \frac{1}{x^2} = ? \)

By calculator or by hand, the graph of \( f(x) = \frac{1}{x^2} \) is...
\[ \lim_{x \to 0^-} \frac{1}{x^2} = +\infty, \quad \lim_{x \to 0^+} \frac{1}{x^2} = +\infty \Rightarrow \text{right and left hand limits are equal and they equal } +\infty \]

\[ \lim_{x \to 0} \frac{1}{x^2} = +\infty \]

\[ \lim_{x \to 1^-} \frac{1}{x-1} = ? \quad \text{and} \quad \lim_{x \to 1^+} \frac{1}{x-1} = ? \]

Graph of \( f(x) = \frac{1}{x-1} \):

\[ \lim_{x \to 1^-} \frac{1}{x-1} = -\infty \]

\[ \lim_{x \to 1^+} \frac{1}{x-1} = +\infty \]
3. \( \lim_{x \to 0^+} \ln x = ? \)

Graph of \( f(x) = \ln x (= \log_e x) \):

\[ y = \ln x \]

\[ \lim_{x \to 0^+} \ln x = -\infty \]

**Calculator:**
- \( x_{\text{Min}} = 0.1 \)
- \( x_{\text{Max}} = 10 \)
- \( y_{\text{Min}} = -10 \)
- \( y_{\text{Max}} = 5 \)

But you must know that the y-axis is a vertical asymptote and \( \text{Domain} = \{ x \mid x > 0 \} \).
 CATEGORY 2.7 \[ \lim_{x \to +\infty} (\text{or } -\infty) f(x) = L \]

"As \( x \) gets larger and larger (+\( \infty \)) (or more and more negative (-\( \infty \))), \( f(x) \) approaches the finite value \( L \)."

This limit does exist, since \( L \) is a single finite value.

**Examples**

1. \( \lim_{x \to -\infty} \frac{1}{x} = \) ?

Graph of \( f(x) = \frac{1}{x} \):

\[ \lim_{x \to -\infty} \frac{1}{x} = [\text{Diagram}] \]
2. (HW Exercise 20, p.140.)
\[
\lim_{t \to \infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3} = ?
\]

We will do some ALGEBRA and use the fact that

For any positive integer \( n \) and any constant \( c \)
\[
\lim_{t \to \infty} \frac{c}{t^n} = 0 \quad \text{and} \quad \lim_{t \to -\infty} \frac{c}{t^n} = 0.
\]

\[
\lim_{t \to \infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3} = \lim_{t \to \infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3} \cdot \frac{\frac{1}{t^3}}{\frac{1}{t^3}}
\]

\[
= \lim_{t \to \infty} \frac{7t^3 + 4t \cdot \frac{1}{t^3}}{2t^3 - t^2 + 3 \cdot \frac{1}{t^3}}
\]

\[
= \lim_{t \to \infty} \frac{7 + 4 \cdot \frac{1}{t^2}}{2 - \frac{1}{t} + 3 \cdot \frac{1}{t^2}} = \frac{7 + 0}{2 - 0 + 0}
\]

\[
= \frac{7}{2}
\]
\[ \lim_{x \to +\infty} f(x) = +\infty \text{ (or } -\infty) \]

"As \( x \) gets larger and larger (or more and more negative), so does \( f(x) \)."

This limit does not exist.

Examples:

1. \( \lim_{x \to +\infty} x^3 = ? \)

Graph of \( f(x) = x^3 \):

\[ \lim_{x \to +\infty} x^3 = +\infty \]
2. \( \lim_{x \to +\infty} \ln x = \) ?

Graph of \( f(x) = \ln x \):

\[
\lim_{x \to +\infty} \ln x = +\infty
\]

3. \( \lim_{x \to +\infty} e^x = ? \)

Graph of \( f(x) = e^x \):

\[
\lim_{x \to +\infty} e^x = +\infty
\]
Lecture

Section 2.6. Tangents, Velocities, and Other Rates of Change.

This section refers to Section 2.1, which we skipped. The material in Section 2.1 is essentially covered, in a formal way, in this section.

**Question:** How does one measure speed at an instant in time?

The answer to this is related to the "derivative," introduced in the next section.

**Example.** (See HW Exercise 16, p. 150.)

If an arrow is shot upward on the moon with a velocity (= speed + direction) of 64 m/s at time t = 0, its height (in meters) after t seconds is given by \( H = 64t - 16t^2 \).

(a) Find the velocity of the arrow after 1 s.
We know that velocity = 0 when t = 2, since the arrow is not moving for an instant.

The graph of $H = 64t - 16t^2 = -16t^2 + 64t$ is an upside down parabola with the t-intercepts $t = 0$ and $t = 4$.

We know velocity = 64 when $t = 0$, since that is given.

We also know how to compute the "average velocity" of the arrow from time $t = 0$ to $t = 1$:

$$\text{average velocity from } t = 0 \text{ to } t = 1 = \frac{\text{change in distance}}{\text{change in time}} = \frac{48 - 0}{1 - 0} = 48$$
How do we compute the velocity at $t = 1$?

Let us try to compute this velocity at an instant in time, just as we would compute average velocity over an interval of time:

$$\text{Instantaneous velocity at } t = 1 = \frac{48 - 48}{1 - 1} = \frac{0}{0}$$

Nonsensical result!

But maybe if we take some sort of limit we will get an answer.

Terminology, Notation, and Definitions

(Instantaneous) Velocity

= speed (magnitude) + direction (up/down, right/left)

refers to speed (with a direction) at an instant in time

Average Velocity

= speed (with a direction) over an interval of time
Let \( s(t) \) = distance traveled as a function of time.

Then

\[
\text{average velocity over the interval } a \leq t \leq b \quad \text{is} \quad \frac{\text{change in distance}}{\text{change in time}} = \frac{s(b) - s(a)}{b - a}
\]

Now, again, defining instantaneous velocity in a similar way is fruitless:

\[
\text{instantaneous velocity at } \quad \text{is} \quad \frac{\text{instantaneous velocity over the interval } a \leq t \leq a}{a - a} = \frac{0}{0}
\]

So, we do what we did in Section 2.2 when we plugged \( x = a \) into \( f(x) \) and obtained \( f(a) = 0 \). We see what value \( \frac{s(a) - s(a)}{t - a} \) approaches as \( t \) approaches \( a \). So, we define instantaneous velocity this way:
\[
\text{instantaneous velocity at } t = a = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}
\]

Graphical Interpretations of
\text{Average and Instantaneous Velocity}
Let $x$ get closer and closer to $a$ as the distance $h$ approaches 0.

Average velocity over the interval $a \leq t \leq x$ is given by:

$$\frac{s(x) - s(a)}{x - a}$$

or

$$\frac{s(a+h) - s(a)}{h}$$
= Slope of secant through P and Q

\[
\text{Instantaneous velocity at } t = a \quad \left\{ \begin{array}{l}
\lim_{{x \to a}} \frac{s(x) - s(a)}{x - a} \\
\lim_{{h \to 0}} \frac{s(a+h) - s(a)}{h}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{l}
\text{Limit of the slopes of the secants through P and Q as Q approaches P} \\
\text{Slope of tangent through P (or at x = a)}
\end{array} \right.
\]

\[\dot{s}, \text{ Ave. Vel. } = \text{slope of secant} \]
\[\text{Inst. Vel. } = \text{slope of tangent}\]
Example. (HW Exercise 16, p. 150.)

If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after $t$ seconds is given by

$$H = 58t - 0.83 t^2.$$ 

(a) Find the velocity of the arrow after 15.
(b) Find the velocity of the arrow when $t = a$.
(c) When will the arrow hit the moon?
(d) With what velocity will the arrow hit the moon?

(c) Let $v(t) = \text{(instantaneous)}$ velocity at time $t$.

$$v(t) = \frac{58t - 0.83t^2}{t - 1}$$

$$= \frac{\lim_{t \to 1} 58(t) - 5(1)}{t - 1}$$

$$= \lim_{t \to 1} \frac{(58t - 0.83t^2) - [58(1) - 0.83(1)^2]}{t - 1}$$

$$= \lim_{t \to 1} \frac{58t - 0.83t^2 - 58 + 0.83}{t - 1}$$

$$= \lim_{t \to 1} \frac{58(t-1) - 0.83(t^2-1)}{t - 1}$$
\[ \lim_{t \to 1} \frac{58(t-1) - 0.83(t+1)(t+1)}{t-1} = \lim_{t \to 1} \left[ 58 - 0.83(t+1) \right] = 58 - 0.83(1+1) = 56.34 \text{ m/s} \]

\[ V(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \text{ instead of } \frac{s(t)}{t-a}, \text{ use } (a+h) - a = h \text{ instead of } \frac{t-a}{t} \]

\[ s(t) = 58t - 0.83t^2 \Rightarrow \]

\[ s(a+h) = 58(a+h) - 0.83(a+h)^2 \]

\[ \lim_{h \to 0} \left[ 58(a+h) - 0.83(a+h)^2 \right] - \left[ 58a - 0.83a^2 \right] \]

\[ = \lim_{h \to 0} \frac{58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2}{h} \]

\[ = \lim_{h \to 0} \frac{58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2 + 0.83a^2}{h} \]
\begin{align*}
&\lim_{h \to 0} \frac{58h - 1.66ah - 0.83h^2}{h} \\
&= \lim_{h \to 0} \frac{h(58 - 1.66a - 0.83h)}{h} \\
&= \lim_{h \to 0} (58 - 1.66a - 0.83h) \\
&= 58 - 1.66a - 0.83(0) \\
&= 58 - 1.66a \quad \text{m/s}
\end{align*}

(c) When the arrow hits the moon, distance \( H \) of the arrow from the moon is 0. So, set \( H = 0 \) and solve for \( t \):

\[ H = 0 \Rightarrow 58t - 0.83t^2 = 0 \]
\[ \Rightarrow t (58 - 0.83t) = 0 \]
\[ \Rightarrow t = 0 \quad \text{or} \quad t = \frac{58}{0.83} \]

\( \uparrow \)

\( \text{When the arrow LEAVES the moon} \quad \text{This must be when the arrow HITS the moon} \)
The velocity with which the arrow hits the moon, which is at time $t = \frac{58}{0.83} \text{ s}$, according to Part (c), can be given by the formula obtained in Part (b):

$$v(a) = 58 - 1.66a$$

$$v\left(\frac{58}{0.83}\right) = 58 - 2\left(\frac{58}{0.83}\right)$$

$$= 58 - 2(58)$$

$$= -58 \text{ m/s}$$

negative since direction of arrow is DOWN.
Let \( f(x) \) be any function with graph. Let \( a \) be a point on the graph and consider the secant line through points \( (a, f(a)) \) and \( (a+h, f(a+h)) \).

**Terminology:**

\[
\frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}
\]

Called the average rate of change of \( f \) over the interval \( a \leq t \leq x \) (or \( a \leq t \leq a+h \)).
or (2) difference quotient of f

or (3) slope of the secant through the points a and x = a + h

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}; \text{ Called}
\]

(1) instantaneous rate of change of f at x = a

or (2) slope of tangent to curve y = f(x) at x = a

or (2) slope of the curve y = f(x) at x = a

---

Examples.

1. (HW Exercise 7, p. 149.)

Find the equation of the tangent line to the curve

\[ y = \sqrt{x^3} \text{ at the point } (1,1). \]
Set \( f(x) = \sqrt{x^2} \) and \( a = 1 \). Then

\[
\text{Slope of tangent at } x = 1 = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}
\]

\[
M_{\text{tan}} = \lim_{x \to 1} \frac{\sqrt{x^2} - \sqrt{1}}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

TRICK: Multiply top and bottom by \( \sqrt{x^2} + 1 \)

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1 + \sqrt{x^2} + 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{x - 1}
\]

\[
= \frac{\sqrt{1} + 1}{1 + 1} = \frac{1}{2}
\]
1. \( m_{tan} = \frac{1}{2} \) \( (x_1, y_1) = (1, 1) \)

\[
\begin{align*}
    y - y_1 &= m_{tan} (x - x_1) \\
    y - 1 &= \frac{1}{2} (x - 1)
\end{align*}
\]

\[
\begin{align*}
    y - 1 &= \frac{1}{2} x - \frac{1}{2} \\
    y &= \frac{1}{2} x + \frac{1}{2}
\end{align*}
\]

2. (HW Exercise 8, p. 149,)

Find the equation of the tangent line to the curve

\[
y = \frac{x}{1-x} \text{ at the point } (0, 0).
\]

Set \( f(x) = \frac{x}{1-x} \) and \( a = 0 \). Then

\[
\text{Slope of tangent line at } a = 0 = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = m_{tan}
\]
\[ m_{\tan} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \]

\[ f(x) = \frac{x}{1-x} \]
\[ f(0+h) = f(h) = \frac{h}{1-h} \]
\[ f(0) = \frac{0}{1-0} = 0 \]

\[ = \lim_{h \to 0} \frac{\frac{h}{1-h} - 0}{h} \]
\[ = \lim_{h \to 0} \frac{h}{1-h} \]
\[ \frac{1}{1-h} \div h = \frac{h}{1-h} \cdot \frac{1}{h} \]
\[ \lim_{h \to 0} \frac{1}{1-h} \]
\[ = \frac{1}{1-0} = \frac{1}{1} = 1 \]

2/27/01
Tues.

Lecture

\[ m_{\tan} = 1 \quad (x_1, y_1) = (0, 0) \]
\[ y - y_1 = m_{\tan} (x - x_1) \Rightarrow \]
\[ y - 0 = 1 \cdot (x - 0) \Rightarrow y = x \]
Section 2.7. Derivatives.

Recall:

\[ f(x) \]

The tangent at \( x = a \)

\[ y = f(x) \]

\[ y' = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

**Inst. rate of change**

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

Defn. The instantaneous rate of change of \( f \) at \( x = a \) is called the derivative of \( f \) at \( x = a \) and is denoted by \( f'(a) \).
So,

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

= \text{slope of tangent line to curve}
\[ y = f(x) \text{ at } x = a \]

Note: It is convention to say that

\[ \text{slope of tangent line} \]
\[ \text{to curve } y = f(x) \text{ at } x = a \]

= \text{slope of curve}
\[ y = f(x) \text{ at } x = a \]
Examples. (Derivative = slope of tangent)

1. (HW Exercise 2, p. 156.)

For function $f$ with graph

$y = f(x)$

arrange following numbers in increasing order (smallest to largest):

$0 \ f'(2) \ f(3) - f(2) \ \frac{1}{2}[f(4) - f(2)]$
Secant line thru $x=2$ and $x=3$ with slope $\frac{f(3)-f(2)}{3-2} = f'(2) > 0$

Tangent line at $x=2$ with slope $f'(2) > 0$

Secant line thru $x=2$ and $x=3$ with slope $\frac{f(3)-f(2)}{3-2} = \frac{1}{2} [f(3) - f(2)] > 0$

$0 < \frac{1}{2} [f(3) - f(2)] < f(3) - f(2) < f'(2)$
If \( g(x) = 1 - x^3 \), find \( g'(0) \) and use it to find the equation of tangent line to curve \( y = 1 - x^3 \) at point \((0, 1)\) \((x = 0)\).

\[
g'(0) = m_{\text{tan}} = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h}
\]

\[
= \lim_{h \to 0} \frac{(1-h^3) - (1-0^3)}{h} = \lim_{h \to 0} \frac{-h^3}{h} = \lim_{h \to 0} -h^2 = -(0)^2 = 0.
\]

\[
\therefore \quad y - y_1 = m_{\text{tan}} (x-x_1) \implies y - 1 = 0 \cdot (x - 0) \implies (0, 1)
\]

\[
y - 1 = 0 \implies y = 1
\]

\[
\frac{y}{y} \to \frac{1}{1} \to \frac{y}{y} = 1
\]

\[
y = 1 - x^3 \quad \text{(tangent line at } x=0)\]
Example. (Finding \( f' \) from the formula for \( f' \))

(HW Exercise 18', p. 156.)

The limit \( \lim_{h \to 0} \frac{(2+h)^3 - 8}{h} \) represents the derivative of some function \( f \) at some point \( a \).

State \( f \) and \( a \).

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

\[
= \lim_{h \to 0} \frac{(2+h)^3 - 8}{h}
\]

\[
f(a+h) = (2+h)^3 \quad \Rightarrow \quad \frac{a=2}{f(a+h) = f(2+h)} \quad \Rightarrow \quad \frac{f(x) = x^3}{f(x) = (2+h)^3}
\]

\[
f(a) = 8 \quad \Rightarrow \quad f(a) = f(2) = 2^3 = 8
\]
Example. (Derivative = rate of change of \( f(x) \) with respect to \( x \))

3/2/01 (HW Exercise 25, p. 157,)

- **Lecture**
  - Cost of producing \( x \) ounces of gold from a new gold mine is \( C = f(x) \) dollars.
  - (a) What is the meaning of \( f'(x) \)?
  - What are its units?
  - (b) What does \( f'(800) = 19 \) mean?
  - (c) **Skip**

\[
\begin{align*}
(a) \quad f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\text{change in cost}}{\text{change in \# ounces produced}} \\
&= \lim_{x \to a} \frac{\Delta C}{\Delta x} \\
&= \frac{\Delta C}{\Delta x} \quad \text{just after (or before) \( x \) is a} \\
& \quad \text{rate of change of cost } C = f(x) \text{ with} \\
& \quad \text{respect to \# ounces } x
\end{align*}
\]
\[ x = a \quad \text{if} \quad f'(a) \]

After \( x \) ounces of gold have been produced, the rate of change of cost with respect to the number of ounces of gold produced is \( f'(x) \).

So, Units of \( f'(x) = \) Units for \( \frac{\Delta C}{\Delta x} = \) dollars/ounce

\[(b) \quad f'(800) = 17 \quad \text{means:} \quad \text{After 800 ounces of gold have been produced, the rate at which cost is increasing is} \quad \$17/\text{per ounce}. \]

\[(f'(800) = -14 \quad \text{means:} \quad \text{After 800 ounces produced, rate at which cost decreasing is} \quad \$17/\text{per ounce}. \)\]
The cost to produce the 801st ounce of gold is \( \$17 \) (or \( f'(800) \)).

In this context, \( f'(800) \) is called the marginal cost.
Example. (Can approximate derivative from table of values of \( f(x) \) versus \( x \))

(HW Exercise 30, p. 157.)

t = time

\( E(t) = \) life expectancy at birth (years) of male born in the year \( t \) in the U.S.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>48.3</td>
</tr>
<tr>
<td>1910</td>
<td>51.1</td>
</tr>
<tr>
<td>1920</td>
<td>55.2</td>
</tr>
<tr>
<td>1930</td>
<td>57.4</td>
</tr>
<tr>
<td>1940</td>
<td>62.5</td>
</tr>
<tr>
<td>1950</td>
<td>65.6</td>
</tr>
<tr>
<td>1960</td>
<td>66.6</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1970</td>
<td>71.8</td>
</tr>
</tbody>
</table>

Interpret and estimate \( E'(1910) \) and \( E'(1950) \).
\[ E'(t) = \lim_{t \to a} \frac{E(t) - E(a)}{t - a}, \quad \text{where can have } t > a \]
\[ \text{or } t < a \]
\[ \approx \frac{E(t) - E(a)}{t - a} \]

\[ E'(1910): \]
\[ t = 1920 \Rightarrow E'(1910) \approx \frac{E(1920) - E(1910)}{1920 - 1910} = \frac{55.2 - 51.1}{10} = 0.41 \]
\[ t = 1900 \Rightarrow E'(1910) \approx \frac{E(1900) - E(1910)}{1900 - 1910} = \frac{48.3 - 51.1}{-10} = 0.28 \]

Take average of these and get even better estimate of \( E'(1910) \):
\[ E'(1910) \approx \frac{1}{2} (0.41 + 0.28) = 0.345 \]
$E'(1910) \approx 0.345$ means:

After 1910, the rate at which life expectancy at birth was increasing was about 0.345 years per year.
Lecture

Section 2.8. The Derivative as a Function.

Function = Domain + Rule

\[ f(x) = \frac{1}{x^2} \]

what \( f \) does to \( x \)

E.g., \( f(x) = \sqrt{x} \)

Domain = \( \{ x \mid x \geq 0 \} \)

Rule = \( \sqrt{\cdot} \)

Recall the ("\( x \)" d) LIMIT DEFINITION of a derivative:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

"Derivative of \( f(x) \) at the point \( x = a \)"

Now, we can replace the "\( a \)" by "\( x \)" and write

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

"Derivative of \( f(x) \)"
E.g., let \( f(x) = x^2 + 1 \).
Find \( f'(2) \) by first finding \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
f(x) = x^2 + 1
\]

\[
f(\Box) = (\Box)^2 + 1
\]

\[
f(x+h) = (x+h)^2 + 1 = x^2 + 2hx + h^2 + 1
\]

\[
= \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 + 1) - (x^2 + 1)}{h}
\]

\[
= \lim_{h \to 0} \frac{2hx + h^2}{h}
\]

\[
= \lim_{h \to 0} \frac{h(2x+h)}{h}
\]

\[
= \lim_{h \to 0} (2x + h)
\]

\[
\text{leave } x \text{ by 0 above!}
\]

\[
= 2x + 0
\]

\[
= 2x
\]

\( f'(x) = 2x \)

\( f'(x) \) is a function where it is a rule acting on \( x \) (multiply by 2).

Then just plug \( x = 10 \) into \( f'(x) = 2x \):

\[
f'(2) = 2(10) = 20
\]
So, \( f'(x) = 2x \) is a function:

- One can plug any value of \( x \) into \( f'(x) \) and obtain a value of \( f'(x) \), e.g., \( f'(4) = 2(4) = 8 \).
- One can draw its graph.
Graphical Way of Obtaining \( f'(x) \)

We just found the FORMULA of \( f'(x) \) from the FORMULA of \( f(x) \) using the limit definition of a derivative.

Now we will find the GRAPH of \( f'(x) \) from the GRAPH of \( f(x) \) using the fact that

\[
f'(a) = \text{slope of tangent line to curve } y = f(x) \text{ at point } x = a.
\]

Example. (HW Exercise 1, p 168)

Use given graph to estimate value of each derivative. Then sketch graph of \( f' \).

\[
\begin{align*}
(a) & \, f'(1) & (b) & \, f'(2) \\
(c) & \, f'(3) & (d) & \, f'(4)
\end{align*}
\]
\[ f'(1) : \quad f'(1) \approx \frac{3}{2} = 1.5 \]

\[ f'(2) : \quad f'(2) \approx \frac{3}{4} = 0.75 \]
\[
f'(3) = \begin{array}{c}
\text{3} \\
\text{0} \\
\end{array} \\
f'(3) \approx -\frac{3}{4} = -0.75
\]

\[
f'(4) = \begin{array}{c}
\text{2} \\
\text{0} \\
\end{array} \\
f'(4) \approx -\frac{3}{4} = -0.75
\]

Plot the points \((1, -1.5), (2, 0.75), (3, -0.75), (4, -0.5)\) and connect the points with a smooth line.
\[ y = f'(x) \]

\[ y' = f''(x) \]
RULES OF THUMB TO QUICKLY SKETCH $f'(x)$ FROM GRAPH OF $f(x)$:

1. A peak or valley in the graph of $f(x)$ corresponds to the graph of $f'(x)$ crossing the $x$-axis.

   - Eq. $y = f(x)$
   - $y' = f'(x)$

2. If $f(x)$ graph increases, $f'(x)$ graph above the $x$-axis.

   - $f(x)$ graph decreasing, $f'(x)$ graph below the $x$-axis.
(3) Pt. between peak and valley in $f(x)$ graph $\Rightarrow$ valley in $f'(x)$ graph

Pt. between valley and peak in $f(x)$ graph $\Rightarrow$ peak in $f'(x)$ graph

E.g. $y = f(x)$ or $y' = f'(x)$

$y = f(x)$ or $y' = f'(x)$
Example. \( (\text{HW Exercise 3, p.168}) \)

Match the graph of each function in (a)-(d) with the graph of its derivative in I-IV.

\[ f(x) \] \(#s:\)

(a) \[ \begin{align*}
\text{Graph} \quad & \quad \text{Graph} \\
\quad & \quad \\
\end{align*} \]

(b) \[ \begin{align*}
\text{Graph} \quad & \quad \text{Graph} \\
\quad & \quad \\
\end{align*} \]

(c) \[ \begin{align*}
\text{Graph} \quad & \quad \text{Graph} \\
\quad & \quad \\
\end{align*} \]

(d) \[ \begin{align*}
\text{Graph} \quad & \quad \text{Graph} \\
\quad & \quad \\
\end{align*} \]
\[ f'(x) \leq 0 : \]

\[ I \quad y' \]

\[ II \quad y' \]

\[ III \quad y' \]

\[ IV \]
(b) $f(x)$

- Slope abruptly changes from positive to negative
- Slope = constant $> 0$
- Slope = constant $< 0$

IV $f'(x)$

- $y'$
- $x$
(c) $f(x)$

Slopes $< 0$ but approaching $0$ as $x \to -\infty$

Slopes $> 0$ but approaching $0$ as $x \to +\infty$

$I \ f'(x)$

Slope $= 0$ at $x = 0$
pt between valley and peak will be peak in graph of $f'(x)$
Differentiability of \( f(x) \)

\( f(x) \) does not always have a derivative at a point \( x = a \) or for \( x \) in general.

**Defn.** \( f(x) \) is said to be **differentiable at** \( x = a \) (or have a derivative at \( x = a \)) if the limit

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

exists.

**Defn.** \( f(x) \) is said to be **differentiable on the interval** \( a < x < b \) if \( f(x) \) is differentiable at every \( x \) in the interval.
Sometimes, we can tell if $f(x)$ is differentiable at some point $x = a$ simply by looking at its graph rather than by evaluating the limit

$$ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. $$

Graphically, there are **four basic ways for $f(x)$ to be undifferentiable at a point $x = a$.**

**Examples**

1. If $f(x)$ is undefined at $x = a$.

   *E.g.,* $f(x) = \frac{1}{x}$

   ![Graph of f(x) undefined at x = a](image)

   $f(a) = \pm \infty \Rightarrow f(a)$ undefined

   $\therefore f'(a)$ does not exist
2. \( f(x) \) defined at \( x=a \)  
   \( \text{but } f(x) \text{ not cont. at } x=a \)

   \( E.g. \)  
   \[ f(x) = \begin{cases} 
   -1, & x < 0, \\
   1, & x \geq 0 
   \end{cases} \]

3. \( f(x) \) defined at \( x=a \)  
   \( f(x) \) cont. at \( x=a \)  
   \( \text{but there are 2 tangent lines at } x=a \)

   \( E.g. \)  
   \[ f(x) = |x| = \begin{cases} 
   -x, & x < 0, \\
   x, & x \geq 0 
   \end{cases} \]
4. \( f(x) \) defined at \( x = a \)
   - \( f(x) \) cont. at \( x = a \)
   - but tangent line is vertical at \( x = a \)

E.g., \( f(x) = x^{1/3} \)

\[ \begin{align*}
\text{1. } f(c) = 0 & \implies f(x) \text{ defined} \\
\text{2. } f(x) \text{ cont. at } x = 0 \text{ by graph} \\
\text{3. BUT tangent line at } x = 0 \\
\text{is vertical and so has slope } = +\infty = \text{undefined} \\
\text{4. } f'(c) \text{ does not exist}
\end{align*} \]
Notation and Derivatives of Derivatives

Notation for the derivative.

\[ f'(x) \quad y' \quad \frac{df}{dx} \quad \frac{dy}{dx} \quad \frac{d}{dx}(f(x)) \]

- Comes from \( f'(a) \approx \frac{df}{dx} \approx \frac{dy}{dx} \)
- Use in Ch. 3 when coming up with formula for derivative using special rules, applied to formula for \( f'(a) \)

Notation for the derivative at \( x = a \).

\[ f'(a) \quad \frac{df}{dx}\bigg|_{x=a} \quad \frac{dy}{dx}\bigg|_{x=a} \quad \frac{d}{dx}(f(x))\bigg|_{x=a} \]
New notation will help us to define these.

\[ f(x) \quad \text{(ZEROth DERIV. of } f(x)) \]
\[ \frac{d}{dx}(f(x)) = f'(x) \quad \text{FIRST DERIV. of } f(x) \]
\[ \frac{d}{dx}(f'(x)) = f''(x) \quad \text{SECOND DERIV. of } f(x) \]
\[ \frac{d}{dx}(f''(x)) = f'''(x) \quad \text{THIRD DERIV. of } f(x) \]
\[ \frac{d}{dx}(f'''(x)) = f^{(4)}(x) \quad \text{FOURTH DERIV. of } f(x) \]

ETC.
Example (HW Exercise 39, p. 171.)

If \( f(x) = 2x^3 - x^2 \), find \( f'(x) \), \( f''(x) \), \( f'''(x) \), \( f^{(4)}(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{2(x+h)^3 - (x+h)^2 - (2x^3 - x^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{2(x^3+3hx^2+3h^2x+h^3) - (2x^3-x^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{6x^2 + 6hx + 3h^2 - x^2 - 3hx^2 - 3h^2x - h^3}{h}
\]

\[
= \lim_{h \to 0} \frac{4hx + 2h^2 - 3hx^2 - 3h^2x - h^3}{h}
\]

\[
= \lim_{h \to 0} \frac{h(4x + 2h - 3hx - 3h^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{4x + 2h - 3hx - 3hx - h^2}{h}
\]

\[
= 4x - 2(0) - 3(0)x - (0)^2
\]

\[
= 4x - 3x^2
\]

\[
f'(x) = 4x - 3x^2
\]
\[ f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \]

\[ f'(x) = 4(x) - 3(x)^2 \]

\[ \lim_{h \to 0} \frac{4(x+h) - 3(x+h)^2 - (4x - 3x^2)}{h} \]

\[ = \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6hx - 3h^2 - 4x + 3x^2}{h} \]

\[ = \lim_{h \to 0} \frac{4h - 6hx - 3h^2}{h} \]

\[ = \lim_{h \to 0} 4 - 6x - 3 \]

\[ = 4 - 6x \]

\[ f''(x) = 4 - 6(x) \]

Etc.
CHAPTER 3 Differentiation.

3/11/01

We will now learn about different RULES OF DIFFERENTIATION (= rules of finding derivatives of functions). These actually come from

1) the limit definition of derivative

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

2) the limit laws (Sec. 2.3), e.g.,

\[ \lim_{x \to 1} (5x^2 + 2x^3) = \lim_{x \to 1} 5x^2 + \lim_{x \to 1} 2x^3 = 5 \cdot \lim_{x \to 1} x^2 + 2 \cdot \lim_{x \to 1} x^3 = 5 \cdot 1^2 + 2 \cdot 1^3 = 7 \]

We will set aside the limit definition of derivative and start memorizing formulas and rules that come from it.
Section 3.1. Derivatives of Polynomials and Exponential Functions.

Notation: If we are given, say, the function \( f(x) = x^2 \), we will write for its derivative

\[
f'(x) = \frac{d}{dx} (x^2) \quad (\text{or } \frac{d}{dx} x^2)
\]

---

**Power Rule**

Let \( f(x) = x \). Then

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h}
\]

\[
= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 = 1 \cdot x^0
\]
\[ f(x) = x^2. \] Then

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x. \]

\[ f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx} \left( \frac{d f(x)}{dx} \right) = \frac{d}{dx} (2x) = 2. \]

\[ f'''(x) = \frac{d^3 f(x)}{dx^3} = \frac{d}{dx} \left( \frac{d^2 f(x)}{dx^2} \right) = \frac{d}{dx} (2) = 0. \]

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x. \]

It can be shown that, in general,

\[ \frac{d}{dx} (x^n) = nx^{n-1}, \quad n \text{ any real number!} \]

\[ \text{POWER RULE} \]

**Examples:**

1. \[ \frac{d}{dx} (x^e) = e \cdot x^{e-1} \quad (e \approx 2.71828...) \]

2. \[ \frac{d}{dx} \left( \sqrt{x} \right) = \frac{d}{dx} (x^{1/2}) = \frac{1}{2} \cdot x^{1/2 - 1} = \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}} \]
Recall: \( x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m \).

\( E.g., \) \( x^{1/2} = \sqrt{x^2} = (\sqrt{x})^2 \).

3. \[
\frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = (-1)x^{-1-1} = \boxed{-x^{-2}} = \boxed{-\frac{1}{x^2}}
\]

**Put in negative exponent form**

Recall: \( x^{-n} = \frac{1}{x^n} \).

4. \[
\frac{d}{dx} (1) = \frac{d}{dx} (x^0) = 0 \cdot x^{0-1} = 0 \cdot x^{-1} = \boxed{0}
\]

Recall: \( x^0 = 1 \).

5. \[
\frac{d}{dx} (x) = \frac{d}{dx} (x^1) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = \boxed{1}
\]

6. \[
\frac{d}{dx} (x^2) = 2x^{2-1} = 2x^1 = \boxed{2x}
\]

7. \[
\frac{d}{dx} (x^3) = 3x^{3-1} = \boxed{3x^2}
\]
Summary of some derivatives you may forget:

Put off #1 and 2 below

\[
\frac{d}{dx}(1) = 0 \quad \frac{d}{dx}(k) = 0, \; k \text{ any constant} \\
E.g., \frac{d}{dx}(x^2) = 0
\]

\[
\frac{d}{dx}(x) = 1 \quad \frac{d}{dx}(kx) = k, \; k \text{ any constant} \\
E.g., \frac{d}{dx}(3x) = 3
\]

---

**Sum/Difference and Constant Multiple Rules**

1. \( \frac{d}{dx}(x^2 + x^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 2x + 3x^2 \)

2. \( \frac{d}{dx} \left( \frac{1}{x} - 1 \right) = \frac{d}{dx} \left( \frac{1}{x} \right) - \frac{d}{dx} (1) \\
   = \frac{d}{dx} (x^{-1}) - \frac{d}{dx} (1) \\
   = -x^{-2} - 0 \\
   = -\frac{1}{x^2} \)
\[ \frac{d}{dx} \left( \frac{1}{2} x^4 \right) = \frac{1}{2} \cdot \frac{d}{dx} (x^4) = \frac{1}{2} (4x^3) = 2x^3 \]

\[ \frac{d}{dx} (2) = \frac{d}{dx} (2.1) = 2 \cdot \frac{d}{dx} (1) = 2(0) = 0 \]

\[ \frac{d}{dx} (3x) = 3 \cdot \frac{d}{dx} (x) = 3(1) = 3 \]

Now we put these rules together to differentiate polynomial functions.

Derivatives of Polynomials

Like on p. 199

Example. (HW Exercise 10, p. 199.)

Differentiate the function

\[ s(t) = t^6 + 6t^2 - 18t^2 + 2t \]

\[ s'(t) = \frac{d}{dt} (t^6 + 6t^2 - 18t^2 + 2t) \]

Notice t instead of x.
\[-180\]  
\[\frac{d}{dt} (t^9) + \frac{d}{dt} (t^7) - 6 \frac{d}{dt} (t^2) + \frac{d}{dt} (2t)\]  
\[= \frac{d}{dt} (t^9) + 7 \frac{d}{dt} (t^7) - 18 \frac{d}{dt} (t^2) + 2 \frac{d}{dt} (t)\]  
\[= 8t^7 + 6(7t^6) - 18(2t^1) + 2(1)\]  
\[= 8t^7 + 42t^6 - 36t + 2\]  

---  

**Derivatives of Non-Polynomials**  

Like \textit{Ex. 14, p. 194.}  

Example. (HW Exercise 14, p. 199.)  

Differentiate the function  
\[H(t) = \sqrt[3]{t^7} (t + 2)\]  

First rewrite \(H(t)\):  
\[H(t) = \sqrt[3]{t^7} (t + 2) = t^{\frac{7}{3}} (t + 2) = t^{\frac{7}{3}} t^1 + 2 t^{\frac{7}{3}}\]  
\[= t^{\frac{7}{3} + 1} + 2 t^{\frac{7}{3}} = (t^{\frac{4}{3}} + 2 t^{\frac{7}{3}})\]  

Then differentiate:  
\[H'(t) = \frac{d}{dt} \left( t^{\frac{7}{3}} + 2 t^{\frac{7}{3}} \right)\]
\[ \frac{d}{dt} \left( t^{\frac{5}{2}} \right) + \frac{d}{dt} \left( \frac{2}{3} t^{\frac{1}{2}} \right) = \frac{4}{3} t^{\frac{3}{2}} + 2 \left( \frac{1}{3} t^{\frac{1}{2}} \right) - \frac{2}{3} t^{-\frac{3}{2}} \]

\[ = \frac{4}{3} \sqrt{t} + \frac{2}{3} \frac{1}{\sqrt{t}} \]

\[ = \frac{4}{3} \sqrt{t} + \frac{2}{3} \left( \frac{1}{\sqrt{t}} \right)^2 \]
Differentiation Formula of the Exponential Function, \( f(x) = e^x \)

Without Proof

(Sketchy proof in this section. Real proof in Section 3.5.)

\[
\frac{d}{dx} (e^x) = e^x
\]

\( f(x) = e^x \) is the ONLY FUNCTION that is the same as its derivative.

Derivative of \( x^e \) vs. Derivative of \( e^x \)

\[
e^x - 1
\]

\( e^x \) Not \( x e^{x-1} \)

Example. (HW Exercise 6, p. 197,)

Differentiate the function

\[
y = 5e^x + 3
\]

\[
y' = \frac{d}{dx} (5e^x + 3) = \frac{d}{dx} (5e^x) + \frac{d}{dx} (3) = 5 \frac{d}{dx} (e^x) + \frac{d}{dx} (3) = 5e^x + 0 = 5e^x
\]
2. If \( f(x) = e^x \), what is \( f'(0) \)?

\[
f'(0) = \left. \frac{d}{dx}(e^x) \right|_{x=0} = e^x \bigg|_{x=0} = e^0 = 1
\]

New notation

Plug \( x = 0 \) into \( e^x \) after you differentiate.

\[
\text{OR}
\]

1^\text{st}

\[ f'(x) = e^x \]

2^\text{nd}

Plug \( x = 0 \); \( f'(0) = 1 \)
What $f'(x)$ and $f''(x)$ Mean

Graphically

(SEE Section 2.10)

First some definitions. I am going to discuss a little of 2.10 now.

$f(x)$ is said to increase (over an interval) if its graph rises from left to right (over that interval).

E.g.,

\[ y = \ln x \]

\[ y = e^{-x} \]

$f(x)$ is said to decrease (over an interval) if its graph falls from left to right (over that interval).

E.g.,
The bending of a curve is called its concavity:

- Bend upward: chord above curve.
- Bend downward: chord below curve.

\( f(x) \) is said to be **concave up** (over an interval) if its graph bends upward (over that interval).

\[ y = e^x \]

\[ y = e^{-x} \]

\( f(x) \) is said to be **concave down** (over an interval) if its graph bends downward (over that interval).

\[ y = \ln x \]

\[ y = -e^x \]
Now some concepts.

f(x) increasing ⇒ slopes > 0 ⇒ f'(x) > 0

f(x) decreasing ⇒ slopes < 0 ⇒ f'(x) < 0

f(x) neither increasing nor decreasing ⇒ slope = 0 ⇒ f'(x) = 0
\[ f'(x) \text{ increasing} \Rightarrow f''(x) > 0 \text{ and } f(x) \text{ concave up} \]

\[ f'(x) \text{ decreasing} \Rightarrow f''(x) < 0 \text{ and } f(x) \text{ concave down} \]

Slopes getting more positive \(\Rightarrow\) slopes increasing

Slopes getting less positive \(\Rightarrow\) slopes decreasing

Slopes getting more negative \(\Rightarrow\) slopes decreasing

Slopes getting less negative \(\Rightarrow\) slopes increasing
Example. With the constant function $f(x) = 3$, graph is a horizontal line

\[
\begin{array}{c}
\text{No increasing/decreasing, } \\
\text{No bending upward/downward, }
\end{array}
\]

and $f'(x) = 0$, $f''(x) = 0$.

Examples.

1. (HW Exercise 41, p. 200,)

On what interval is the function

\[ f(x) = 1 + 2e^x - 3x \]

increasing?

**Step 1.** Find $f'(x)$.

\[ f'(x) = \frac{d}{dx} (1 + 2e^x - 3x) \]

\[ = \frac{d}{dx} (1) + 2 \frac{d}{dx} (e^x) - 3 \frac{d}{dx} (x) \]
\[-188-\]

\[= 0 + 2e^x - 3(1)\]

\[= 2e^x - 3\]

**STEP 2.** Set \(f'(x) > 0\) and solve for \(x\).

\[f'(x) > 0 \Rightarrow 2e^x - 3 > 0 \Rightarrow 2e^x > 3\]

\[\Rightarrow e^x > \frac{3}{2} \Rightarrow \ln e^x > \ln \left(\frac{3}{2}\right)\]

\[\Rightarrow x > \ln \left(\frac{3}{2}\right) \approx 0.4\]

\[
\begin{array}{c}
\ln \left(\frac{3}{2}\right) \approx 0.4 \\
\end{array}
\]

\[
\left(\ln \left(\frac{3}{2}\right), \infty\right)
\]

\(f(x)\) increasing over this interval
2. (HW Exercise 42, p. 200.)

On what interval is the function

\[ f(x) = x^3 - 4x^2 + 5x \]

concave up?

**HINT:**
- Find \( f'(x) \).
- Then find \( f''(x) \).
- Then set \( f''(x) > 0 \) (means "concave up") and solve for \( x \).

3. (HW Exercise 43, p. 200.)

Find the points on the curve

\[ y = x^3 - x^2 - x + 1 \]

where the tangent is horizontal.

**HINT:**
- Find \( f'(x) \).
- Set \( f'(x) = 0 \) and solve for \( x \).
Lecture

Section 3.2 The Product and Quotient Rules [Without Proof]

Let \( h(x) = x^3 \sqrt{x} \). Then
\[
  h(x) = x^3 x^{1/2} = x^{3 + \frac{1}{2}} = x^{\frac{7}{2}}
\]

and
\[
  h'(x) = \frac{d}{dx} \left( x^{7/2} \right) = \frac{7}{2} x^{\frac{7}{2} - 1} = \frac{7}{2} x^{\frac{5}{2}}
\]

Next, do not "combine" the \( x^3 \) and the \( \sqrt{x} \) into one power function \( x^{7/2} \). Instead view \( x^3 \) and \( \sqrt{x} \) as two separate functions in a product:
\[
f(x) = x^3 \quad \text{and} \quad g(x) = \sqrt{x}.
\]

and use the product rule to find the derivative of the product
\[
f(x)g(x) = x^3 \sqrt{x}.
\]
PRODUCT RULE

\[
\frac{d}{dx} \left[ f(x) g(x) \right] = \left[ \frac{d}{dx} f(x) \right] g(x) + f(x) \left[ \frac{d}{dx} g(x) \right]
\]

\[
(f \cdot g)' = f' \cdot g + f \cdot g'
\]

**ASIDE:** Simplification of Proof in Text

May be an EXTRA CREDIT somewhere

\[
\left[ f(x) g(x) \right]' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x)}{h}
\]

TRICK:

Need something like

\[
\lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{[f(x+h) - f(x)] g(x+h) + f(x) [g(x+h) - g(x)]}{h}
\]
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x) \\
= f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]
\[ h'(x) = \frac{d}{dx} \left( x^3 \cdot \sqrt{x} \right) = \left( \frac{d}{dx} x^3 \right) \cdot \sqrt{x} + x^3 \left( \frac{d}{dx} \sqrt{x} \right) \]

\[ = \left( 3x^2 \right) \cdot \sqrt{x} + x^3 \left( \frac{1}{2x} \right) \]

\[ = 3x^{2.5} + \frac{1}{2}x^{3.5} \]

\[ = 3 \cdot 5^{0.5} + \frac{1}{2} \cdot 5^{1.5} \]

\[ = \left( 3 + \frac{1}{2} \right) \cdot 5^{0.5} \]

\[ = \left( \frac{7}{2} \right) \cdot 5^{0.5} \]

\[ 3 + \frac{1}{2} = \frac{7}{2} \]
NEVER DO THIS!

\[ h'(x) = \frac{d}{dx} \left( x^3 x^{\frac{1}{2}} \right) = \left( \frac{d}{dx} x^3 \right) \left( \frac{d}{dx} x^{\frac{1}{2}} \right) \]

\[ = (3x^2)(\frac{1}{2}x^{-\frac{1}{2}}) \]

\[ = \frac{3}{2} x^{2-\frac{1}{2}} \]

\[ = \frac{3}{2} x^{\frac{3}{2}} \quad \text{WRONG!} \]

\[ 2^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{3}{2} \]

There will be situations, like \( xe^x \), where you cannot "combine" two functions in a product to create a power function or a familiar single function. In this case, the power rule is necessary.
Examples.

1. (HW Exercise 1, p. 206) Find derivative of \( y = (x^2 + 1)(x^3 + 1) \) in two ways:
   (1) Product Rule
   (2) Multiplying out first

\[
y' = \frac{d}{dx} \left[ (x^2 + 1)(x^3 + 1) \right] = (f \cdot g)' = f'g + fg'
\]

\[
= \left[ \frac{d}{dx} (x^2 + 1) \right] (x^3 + 1) + (x^2 + 1) \left[ \frac{d}{dx} (x^3 + 1) \right]
\]

\[
= (2x + 0)(x^3 + 1) + (x^2 + 1)(3x^2 + 0)
\]

\[
= 2x(x^3 + 1) + (x^2 + 1) \cdot 3x^2
\]

\[
= 2x^4 + 2x + 3x^4 + 3x^2
\]

\[
= 5x^4 + 3x^2 + 2x
\]
\( y = (x^2 + 1)(x^3 + 1) = x^5 + x^3 + x^2 + 1 \)

\[
y' = \frac{d}{dx} (x^5 + x^3 + x^2 + 1)
   = 5x^4 + 3x^2 + 2x + 0
   = \boxed{5x^4 + 3x^2 + 2x}
\]

Both answers are the same. \( \checkmark \)
Differentiate \( f(x) = x^2 e^x \)

We cannot multiply out and use the power rule only. We have no choice but to use the product rule.

\[
\begin{align*}
 f'(x) &= \frac{d}{dx} (x^2 e^x) \\
 &= \left[ \frac{d}{dx} (x^2) \right] e^x + x^2 \left[ \frac{d}{dx} (e^x) \right] \\
 &= (2x) e^x + x^2 (e^x) \\
 &= 2x e^x + x^2 e^x \\
 &= xe^x (2 + x)
\end{align*}
\]
Now let \( h(x) = \frac{1+x^3}{x} \). Then
\[
h(x) = \frac{1+x^3}{x} = \frac{1}{x} + \frac{x^3}{x} = x^{-1} + x
\]
and
\[
h'(x) = \frac{d}{dx} \left( x^{-1} + x \right) = \frac{d}{dx} (x^{-1}) + \frac{d}{dx} (x) = (-1)x^{-1-1} + 1 = -x^{-2} + 1
\]

Next, do not "split apart" the
\( \frac{1+x^3}{x} \) into two power functions
\( x^{-1} \) and \( x \).

Instead view \( \frac{1+x^3}{x} \) as two functions
in a quotient:

\[
f(x) = 1 + x^3 \text{ and } g(x) = x
\]

and use the quotient rule to find the derivative of the quotient
\[
\frac{f(x)}{g(x)} = \frac{1 + x^3}{x}
\]
QUOTIENT RULE

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\frac{d}{dx} f(x) g(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}
\]

\[
\left( \frac{f}{g} \right)' = \frac{f' \cdot g - f \cdot g'}{g^2}
\]
So,

\[ h'(x) = \frac{d}{dx} \left( \frac{1 + x^2}{x} \right) \]

\[ = \left[ \frac{d}{dx} (1 + x^2) \right] \frac{x}{x^2} - (1 + x^2) \left( \frac{d}{dx} \frac{1}{x} \right) \]

\[ = \frac{(0 + 2x)x - (1 + x^2)(-1)}{x^2} \]

\[ = \frac{2x^2 - 1 - x^2}{x^2} \]

\[ = \frac{x^2 - 1}{x^2} \]

\[ = \frac{x^2}{x^2} - \frac{1}{x^2} \]

\[ = 1 - \frac{1}{x^2} \]
Examples.

1. (HW Exercise 8, p. 206.)

Differentiate \( f(u) = \frac{1-u^2}{1+u^2} \)

\[
f'(u) = \frac{d}{du} \left( \frac{1-u^2}{1+u^2} \right)
= \frac{\left[ \frac{d}{du} (1-u^2) \right] (1+u^2) - (1-u^2) \left[ \frac{d}{du} (1+u^2) \right]}{(1+u^2)^2}
= \frac{(0-2u)(1+u^2) - (1-u^2)(0+2u)}{(1+u^2)^2}
= \frac{-2u - 2u^3 - 2u - 2u^3}{(1+u^2)^2}
= \frac{-4u}{(1+u^2)^2}
\]
2. (HW Exercise 2, p. 206.)

Find the derivative of

\[ F(x) = \frac{x - 3x^{\frac{3}{2}}}{\sqrt{x}} \]

in two ways:

(1) Quotient Rule

(2) Simplifying first

\[ F(x) = \frac{x - 3x^{\frac{3}{2}}}{\sqrt{x}} = \frac{x - 3x^{\frac{1 + \frac{1}{2}}}{\sqrt{x}}} \]

\[ = \frac{x - 3x^{\frac{3}{2}}}{\sqrt{x}} \]

\[ 1 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{3}{2} \]

(1)

\[ F'(x) = \frac{\left( \frac{d}{dx} \left( x - 3x^{\frac{3}{2}} \right) \right) \sqrt{x} - \left( x - 3x^{\frac{3}{2}} \right) \frac{d}{dx} \left( \sqrt{x} \right)}{\left( \sqrt{x} \right)^2} \]

(2)

\[ F(x) = \frac{x - 3x^{\frac{3}{2}}}{\sqrt{x}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} - 3 \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} \]

\[ = x^{\frac{1}{2}} - 3 \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} = x^{\frac{1}{2}} - 3 \]

\[ 1 - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = \frac{3}{2} \]

\[ \frac{3}{2} - \frac{1}{2} = \frac{1}{2} = 1 \]
\[ F'(x) = \frac{d}{dx} \left( x^{\frac{3}{2}} - 3x \right) = \cdots \]

Answers to (1) and (2) should be the same! Takes algebra to show this through.
3. Let \( f(x) = \frac{e^x}{x} \). Find \( f'(x) \) and \( f''(x) \)

\[
f'(x) = \frac{d}{dx} \left( \frac{e^x}{x} \right) = \frac{\left( \frac{d}{dx} e^x \right) x - e^x \left( \frac{d}{dx} x \right)}{x^2} = \frac{(e^x) x - e^x (1)}{x^2} = \frac{xe^x - e^x}{x^2}
\]

\[
f''(x) = \frac{d}{dx} \left( \frac{xe^x - e^x}{x^2} \right) = \frac{\left[ \frac{d}{dx} \left( xe^x - e^x \right) \right] x^2 - (xe^x - e^x) \left( \frac{d}{dx} x^2 \right)}{(x^2)^2}
\]

\[
= \frac{\left[ \frac{d}{dx} (xe^x) - e^x \right] x^2 - (xe^x - e^x) (2x)}{x^2} = \frac{(xe^x - e^x)}{x^2}
\]

Need **PRODUCT RULE** to evaluate \( \frac{d}{dx} (xe^x) \):

\[
\frac{d}{dx} (xe^x) = \left( \frac{d}{dx} x \right) e^x + x \left( \frac{d}{dx} e^x \right) = (1)e^x + x(e^x) = e^x + xe^x
\]
\[ \frac{\frac{x}{x} - \frac{x^2}{x^2} + \frac{x}{x}}{\left( \frac{x + x}{x} \right) - \frac{x^2}{x} - \frac{x}{x}} = \frac{e^{-x}}{10} \]
4. (HW Exercise 25(a), p. 206.)

Suppose \( f(5) = 1 \)
\( f'(5) = 6 \)
\( g(5) = -3 \)
\( g'(5) = 2 \)

Find the value of \((fg)'(5)\)

\[(fg)'(5) = \frac{d}{dx}[f(x)g(x)]\bigg|_{x=5}
\]

4. **Means: First take the derivative of \(f(x)g(x)\), then plug in \(x = 5\) into your result.**

\[
= f'(x)g(x) + f(x)g'(x)\bigg|_{x=5}
\]

**Product Rule**

\[
= f'(5)g(5) + f(5)g'(5)
\]

\[
= (6)(-3) + (1)(2)
\]

\[
= -16
\]
On what interval is the function \( f(x) = x^3 e^x \) increasing?

**STEP 1.** Find \( f'(x) \) using the PRODUCT RULE:

\[
f'(x) = \frac{d}{dx} (x^3 e^x) = 3x^2 e^x + x^3 e^x
\]

\( (f \cdot g)' = f' \cdot g + f \cdot g' \)

**STEP 2.** Set \( f'(x) > 0 \) and solve for \( x \):

\[
f'(x) > 0 \implies 3x^2 e^x + x^3 e^x > 0
\]

\[
\implies e^x (3x^2 + x^3) > 0
\]

\[
\implies \frac{e^x}{e^x} (3x^2 + x^3) > 0
\]

\[
\implies 3x^2 + x^3 > 0
\]

Can divide by \( e^x \) without changing inequality since \( e^x > 0 \)
\[ 3x^2 + x^3 > 0 \]
\[ x^2(3+x) > 0 \]

**TABLE OF SIGNS**

<table>
<thead>
<tr>
<th>( x^2 )</th>
<th>( 3+x )</th>
<th>( x^2(3+x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ + + + + + +</td>
<td>- - -</td>
<td>- - 0 + + + + + +</td>
</tr>
<tr>
<td>((-3, 0))</td>
<td>((-3, 0))</td>
<td>((0, \infty))</td>
</tr>
</tbody>
</table>

When \( 3 + x = 0 \)  
When \( x^2 = 0 \)

\[ \therefore f'(x) > 0 \text{ on } (-3, 0) \cup (0, \infty) \implies f(x) \text{ is increasing on } (-3, 0) \cup (0, \infty) \]

"[Text says \((-3, \infty)\), where a single point, in this case \(x = 0\), does not matter.]"
Example. (Exercise 35, p. 205.)

On what interval is the function

$$F(x) = x^3 e^x$$

increasing?

Need to find all values of $x$ that make $F'(x) > 0$.

**STEP 1.** Find $F'(x)$ using the Product Rule.

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$F'(x) = \frac{d}{dx} (x^3 e^x)$$

$$= (\frac{d}{dx} x^3) e^x + x^3 (\frac{d}{dx} e^x)$$

$$= 3x^2 e^x + x^3 e^x$$

$$= x^2 e^x (3 + x)$$
**STEP 2.** Set \( F'(x) > 0 \) and solve for \( x \).

\[
F'(x) > 0 \Rightarrow 3x^2 e^x + x^3 e^x > 0
\]

\[
\Rightarrow x^3 e^x > -3x^2 e^x
\]

\[
\Rightarrow x^2 > -3
\]

\[
\Rightarrow x > -\sqrt{3} \quad \text{and} \quad x \neq 0
\]

(As long as \( x \neq 0 \), divide by \( x^2 \).)

When \( x = 0 \),

\[ F'(x) = 0 \Rightarrow \text{tangent horizontal} \]

\[
\Rightarrow F \text{ neither inc. nor dec.}
\]

\[ (-\sqrt{3}, 0) \cup (0, \infty) \]

\( F(x) \) is increasing over this interval.

(Solution manual says \( [-\sqrt{3}, 0) \cup (0, \infty) \), but a single point \( x = 0 \) does not matter.)
Section 3.4. Derivatives of Trigonometric Functions.

**REVIEW OF TRIG FUNCTIONS**

There are six basic trig functions and their inverses:

\[
\begin{align*}
\sin t & \quad \csc t = \frac{1}{\sin t} \\
\cos t & \quad \sec t = \frac{1}{\cos t} \\
\tan t & \quad \cot t = \frac{1}{\tan t}
\end{align*}
\]

\[
\begin{align*}
\sin^{-1} t & = \arcsin t \\
\cos^{-1} t & = \arccos t \\
\tan^{-1} t & = \arctan t
\end{align*}
\]

(t in radians, not in degrees!!)
Notes:

1. \( \sin^{-1} (\sin t) = t \) \( \sin (\sin^{-1} t) = t \)

2. \( \cos^{-1} (\cos t) = t \) \( \cos (\cos^{-1} t) = t \)

3. \( \tan^{-1} (\tan t) = t \) \( \tan (\tan^{-1} t) = t \)
4. $t$ must be in radians when using the derivative formulas for the first six trig functions.

**REASON:** Derivative formula for $\sin t$ is obtained from limit definition of derivative

$$f'(t) = \lim_{h \to 0} \frac{\sin(t+h) - \sin(t)}{h},$$

which, in turn, uses the fact that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

which, in turn, assumes $t$ is in radians.

Derivative formula for $\cos t$ comes from fact that

$$\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t - 1}{t} = 0,$$

Derivative formulas for $\tan t$, $\csc t$, $\sec t$, and $\cot t$ come from derivative formulas for $\sin t$ and $\cos t$ and from using quotient rule.
5. Will use $t$, $x$, $\theta$ (Greek letter "theta"), etc., as the independent variable whose units are radians.
5. IDENTITIES YOU SHOULD KNOW

\( \sin^2 x + \cos^2 x = 1 \) (Fundamental Identity)

\[
\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}
\]

\[
\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x}
\]

\( \tan^2 x + 1 = \sec^2 x \)

3. \( \sin(x + k(2\pi)) = \sin(x) \) \quad \left\{ \begin{array}{l}
k = 0, \pm 1, \pm 2, \ldots
\end{array} \right. \\
\cos(x + k(2\pi)) = \cos(x) \\

e.g., \quad \sin(x + 2\pi) = \sin(x) \\
\cos(x - 2\pi) = \sin(x + (-1)(2\pi)) \\
\sin(x + (2\pi)) = \sin(x) \\
\cos(x + 2\pi) = \cos(x)

e.g., \( \sin x = 1 \) for what values of \( x \)? \( \begin{array}{c}
x = \frac{\pi}{2} + k(2\pi), \quad k = 0, \pm 1, \pm 2, \ldots
\end{array} \)

\( \cos x = -\frac{1}{2} \) for what values of \( x \)?
\[ x = \frac{2\pi}{3} + k(2\pi), \quad k = 0, \pm 1, \pm 2, \ldots \]

(SEE HW Exercise 25, p. 236.)
Derivatives of Trig Functions

**MEMORIZE**

\[
\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x
\]

(x in RADIIANS!)

Do not worry about how these derivatives were derived.

However, know how to derive \( \frac{d}{dx}(\tan x) \) using the Quotient Rule:

\[
\frac{f}{g} = \frac{f'g - fg'}{g^2}
\]

\[
\tan x = \frac{\sin x}{\cos x} \Rightarrow \\
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2} = \frac{(\cos x)\cos x - \sin x(-\sin x)}{(\cos x)^2}
\]
\[
\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x
\]
Also be able to derive:

\[
\frac{d}{dx} \csc x = \frac{d}{dx} \left( \frac{1}{\sin x} \right)
\]

\[
\frac{d}{dx} \sec x = \frac{d}{dx} \left( \frac{1}{\cos x} \right)
\]

\[
\frac{d}{dx} \cot x = \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right)
\]
Examples.

1. (HW Exercise 1, p. 225.)

\[ y = \sin x + \cos x \]

\[ y' = \frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) = \cos x - \sin x \]


2. (HW Exercise 4, p. 225.)

\[ y = e^x \sin x \]

\[ y' = \frac{d}{dx}(e^x \sin x) \]

\[ \text{Use Product Rule: } (fg)' = fg' + f'g \]

\[ = (\frac{d}{dx} e^x) \sin x + e^x \left( \frac{d}{dx} \sin x \right) \]

\[ = e^x \sin x + e^x \cos x \]

\[ = e^x (\sin x + \cos x) \]
\[
y = x \csc x
\]

\[
y' = \frac{d}{dx}(x \csc x) = \frac{d}{dx}(x \cdot \frac{1}{\sin x})
\]

\[
= \frac{d}{dx}\left(\frac{x}{\sin x}\right)
\]

USE THE QUOTIENT RULE: \((f/g)' = \frac{fg' - f'g}{g^2}\)

\[
= \frac{\left(\frac{d}{dx}x\right)\sin x - x \left(\frac{d}{dx} \sin x\right)}{(\sin x)^2}
\]

\[
= \left(1\right) \frac{\sin x - x \cos x}{\sin^2 x}
\]

\[
= \frac{\sin x - x \cos x}{\sin^2 x}
\]

\[
= \frac{\sin x}{\sin^2 x} - \frac{x \cos x}{\sin^2 x}
\]

\[
= \frac{1}{\sin x} - x \cdot \frac{\cos x}{\sin x}
\]

\[
= \csc x - x \csc x \cot x
\]

\[
= \csc x (1 - x \cot x)
\]
4. (HW Exercise 12, p. 226.)

\[ y = x \sin x \cos x \]

\[ y' = \frac{d}{dx} \left[ x \sin x \cos x \right] \]

- USE PRODUCT RULE TWICE:
  \[ (fg)' = f'g + fg' \]

\[ \left[ \frac{d}{dx} (x \sin x) \right] \cos x + x \sin x \left( \frac{d}{dx} \cos x \right) \]

\[ \left[ \frac{d}{dx} (x \sin x) \right] \cos x + x \sin x (-\sin) \]

\[ = \left[ \left( \frac{d}{dx} x \right) \sin x + x \left( \frac{d}{dx} \sin x \right) \right] \cos x - x \sin^2 x \]

\[ = \left[ \sin x + x \cos x \right] \cos x - x \sin^2 x \]

\[ = \sin x \cos x + x \cos^2 x - x \sin^2 x \]
Section 3.5. The Chain Rule

Product Rule: Method for finding derivative of product of 2 functions

\[(f \cdot g)' = f' \cdot g + f \cdot g'\]

Quotient Rule: Method for finding derivative of quotient of 2 functions

\[\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}\]

Chain Rule: Method for finding derivative of composition of 2 functions

\[(f \circ g)'(x) = (f(g(x)))' = \frac{d}{dx}f(g(x))\]
Examples of Compositions of 2 Functions.

1. \( f(g(x)) = \sqrt{1-x^2} \); \( f(x) = \sqrt{x^2} \), \( g(x) = 1-x^2 \)

2. \( f(g(x)) = e^{-4x} \); \( f(x) = e^x \), \( g(x) = -4x \)

3. \( f(g(x)) = (x^3+1)^{100} \); \( f(x) = x^{100} \), \( g(x) = x^3+1 \)

4. \( f(g(x)) = \frac{1}{(1+e^x)^2} \); \( f(x) = \frac{1}{x^2} = x^{-2} \), \( g(x) = 1 + e^x \)

CHAIN RULE

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)
\]

Derivative of outside function \( f(g(x)) \), composed with \( g(x) \)

Derivative of inside function \( g(x) \)
To obtain \( f'(g(x)) \), do the following:

Let \( u = g(x) \) and \( f(u) = f(g(x)) \).

Find \( f'(u) \).

Replace \( u \) by \( g(x) \) to obtain \( f'(g(x)) \).

E. g., \( f(g(x)) = (x^2 + 4)^{50} \)

Here \( g(x) = x^2 + 4 \)

Let \( u = x^2 + 4 \) and \( f(u) = u^{50} \).

Then \( f'(u) = 50u^{49} \) and

\( f'(g(x)) = 50(x^2 + 4)^{49} \).

ALTERNATIVE FORM OF THE CHAIN RULE

\[
\frac{d}{dx} f(g(x)) = \left[ \frac{df}{du} \right]_{u=g(x)} \cdot \left[ \frac{dx}{dx} \right]_{u=g(x)}
\]

\[
\text{derivative of outside function with respect to } u \quad \text{derivative of inside function with respect to } x
\]
Example.
Find the derivative of
\[ F(x) = (x^2+1)^2. \]

Two ways to find \( F'(x) \):

1. First multiply out \((x^2+1)^2\) and then use only the Power Rule.

\[
F(x) = (x^2+1)^2 = (x^2+1)(x^2+1) = x^4 + 2x^2 + 1
\]

\[
F'(x) = \frac{d}{dx} (x^2+1)^2 = \frac{d}{dx} (x^4 + 2x^2 + 1)
\]

\[
= 4x^3 + 4x + 0
\]

\[
= 4x^3 + 4x
\]
(2) Do not multiply out \((x^2+1)^2\) and instead use the CHAIN RULE.

\[ F(x) = (x^2 + 1)^2 = f(g(x)) \]

outside function: \(f(x) = x^2\)
inside function: \(g(x) = x^2 + 1\)

\[ F'(x) = \frac{d}{dx} (x^2 + 1)^2 = \left(\frac{d}{du} u^2\right) \left[ \frac{d}{dx} (x^2 + 1) \right] \]

\[ = (2u)(2x+0) = \left[ 2(x^2+1) \right](2x+0) \quad u=x^2+1 \]

\[ = (2x^2 + 2)(2x) = \boxed{4x^3 + 4x} \]

\[ \checkmark \]

\[ \text{Check mark} \]
The CHAIN RULE can help us derive the formula:

\[
\frac{d}{dx} a^x = a^x \ln a.
\]

**Note:** When \(a = e\), we have

\[
\frac{d}{dx} e^x = e^x \ln e = e^x \quad \checkmark
\]

**Example.**

Find the derivative of

\[ F(x) = 3^x. \]

**Note that**

1. \(e^{\ln y} = y\)
2. \(\ln y^x = x \ln y\)

\[
e^{x \ln 3} = e^{\ln 3^x} = 3^x
\]
\[ F(x) = 3^x = e^{x \ln 3} = e^{(\ln 3)x} = e^{1.1x} \]

where

\[ f(x) = e^x \quad \text{and} \quad g(x) = (\ln 3)x \]

\[ F'(x) = \frac{d}{dx} 3^x \]

\[ = \frac{d}{dx} e^{(\ln 3)x} = u \]

\[ = (\frac{d}{du} e^u) \cdot \left[ \frac{d}{dx} (\ln 3)x \right] \]

\[ = (e^u)(\ln 3) \left( \frac{d}{dx} x \right) \]

\[ = (e^u)(\ln 3) (1) \]

\[ = \left[ e^{(\ln 3)x} \right] (\ln 3) \]

\[ = \left[ 3^x \ln 3 \right] \]
Examples.

\((\text{HW Exercise 1, p. 234.})\) \[ y = (x^2 + 4x + 6)^5 \]

**NOTE:** It would not be convenient to multiply out \((x^2 + 4x + 6)^5\) and then use the **power rule**. Here the **chain rule** is necessary!

\[ y = (x^2 + 4x + 6)^5 = f(g(x)) \]

outside function: \( f(x) = x^5 \)
inside function: \( g(x) = x^2 + 4x + 6 \)

\[ y' = \frac{d}{dx} (x^2 + 4x + 6)^5 = \left( \frac{d}{du} u^5 \right) \left[ \frac{d}{dx} (x^2 + 4x + 6) \right] \]

\[ = (5u^4)(2x + 4 + 0) = [5(x^2 + 4x + 6)^4](2x + 4 + 0) \]

\[ u = x^2 + 4x + 6 \]

\[ = [5(x^2 + 4x + 6)^4](2x + 4) = (10x + 20)(x^2 + 4x + 6)^4 \]
\[ y = \tan(3x) \]

Outside function: \( f(x) = \tan x \)
Inside function: \( g(x) = 3x \)

\[ y' = \frac{d}{dx} \tan(3x) = \left( \frac{d}{dx} \tan u \right) \cdot \left( \frac{d}{dx} 3x \right) \]
\[ = (\sec^2 u) \cdot (3 \cdot \frac{d}{dx} x) = (\sec^2 u) (3 \cdot 1) \]
\[ = 3 \sec^2 (3x) \]

\( u = 3x \)
\[ y = \frac{e^{x}}{x^2} \]

**Lecture**

\[ y = e^{\sqrt{x}} = f(g(x)) \]

**outside function:** \( f(x) = e^x \)

**inside function:** \( g(x) = x^{\frac{1}{2}} \)

\[ y' = \frac{d}{dx} e^{\sqrt{x}} = \left( \frac{d}{du} e^u \right) \cdot \left( \frac{d}{dx} x^{\frac{1}{2}} \right) \]

\[ = (e^u) \left( \frac{1}{2} x^{-\frac{1}{2}} \right) = (e^{\sqrt{x}}) \left( \frac{1}{2} x^{-\frac{1}{2}} \right) \]

\[ u = x^{\frac{1}{2}} \]

\[ = \frac{1}{2} x^{-\frac{1}{2}} e^{\sqrt{x}} \]

\[ = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} \cdot \frac{e^{\sqrt{x}}}{1} \]

\[ = \frac{e^{\sqrt{x}}}{2 \sqrt{x}} \]
4. (HW Exercise 19, p. 234.)

\[ F(y) = \left( \frac{y - 6}{y + 9} \right)^3 \]

\[ F(y) = \left( \frac{y - 6}{y + 9} \right)^3 = f(g(y)) \]

outside function: \( f(y) = y^3 \)
inside function: \( g(y) = \frac{y - 6}{y + 9} \)

\[ F'(y) = \frac{d}{dy} \left( \frac{y - 6}{y + 9} \right)^3 = \left( \frac{d}{du} u^3 \right) \cdot \left[ \frac{d}{dy} \left( \frac{y - 6}{y + 9} \right) \right] = \ldots \]

Use the Quotient Rule.
### Section 3.6. Implicit Differentiation

**Examples of Explicit and Implicit Functions**

<table>
<thead>
<tr>
<th>Explicit Function of $x$</th>
<th>Implicit Function of $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3(x-2)^2 + 5$</td>
<td>$x = 3(y-2)^2 + 5$</td>
</tr>
<tr>
<td>$y = e^{x^2} + 3$</td>
<td>$x = y + e^y$</td>
</tr>
<tr>
<td>$y = \sin x^2$</td>
<td>$y^2 - 2x^2y^2 + 1 = 0$</td>
</tr>
<tr>
<td>$y = \frac{1}{x}$</td>
<td>$\frac{y}{y^2 + x} = 7$</td>
</tr>
</tbody>
</table>

- $y$ is isolated and placed on one side of $\frac{d}{dx}$.
- Expression in $x$ only is on the other side of $\frac{d}{dx}$.
- $y$ is "solved for".
- Usually $x$'s and $y$'s are "mixed together" or $y$ is raised to some power $>1$ and there is more than one $y$ term.
- $y$ cannot be "solved for" or solving for $y$ will...
lead to more than one function, as with
\[ x^2 + y^2 = 1 \Rightarrow \]
\[ y = \pm \sqrt{1-x^2} \Rightarrow \]
\[ y = \sqrt{1-x^2}, \quad y = -\sqrt{1-x^2} \]
\[
\begin{array}{c}
\text{2 different functions}
\end{array}
\]

**NOTES:**

1. With \( x^2 + y^2 = 1 \): \( y \) is an implicit function of \( x \) since one cannot solve for \( y \) explicitly.

\[ x^2 + y^2 = 1 \Rightarrow y^2 = 1-x^2 \]
\[ \Rightarrow y = \pm \sqrt{1-x^2} \]

So \( y \) is not given explicitly, or definitely, or unambiguously.
Either $y$ is the function
\[ y = \sqrt{1 - x^2} \]
with graph

or $y$ is the function
\[ y = -\sqrt{1 - x^2} \]
with graph

2. But with $x^3 + y^3 = 1$; $y$ is an explicit function of $x$ since one can solve for $y$ explicitly
\[
x^3 + y^3 = 1 \Rightarrow y^3 = 1 - x^3 \Rightarrow y = \sqrt[3]{1 - x^3}
\]
So, $y$ is given explicitly as the function

$$y = \sqrt[3]{1-x^3}$$

with graph:  

![Graph of $y = \sqrt[3]{1-x^3}$](attachment:image.png)
**Implicit Differentiation**

**Definition.** Implicit differentiation is the method used to find the derivative of an implicit function of $x$, usually by taking the derivative, $\frac{dy}{dx}$, of both sides of the implicit equation in $x$ and $y$.

**NOTES:**
1. We view $y$ as a function (unknown) of $x$:

$$ y = y(x) $$

2. We then use the chain rule on $y$:

$$ \frac{d}{dx} f(y(x)) = f'(y(x)) \cdot y'(x) $$
Examples.

1. Let $x^2 + y^2 = 4$. Find $y'$.

$y$ is an implicit function of $x$ and $x^2 + y^2 = 4$ is an implicit equation. We can approach this problem in two different ways.

1. **Implicit Differentiation**:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(y) \Rightarrow$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(y) \Rightarrow$$

$$2x + 2yy' = 0 \Rightarrow$$

$$\frac{dy}{dx}(y(x))^2 = 2(y(x))^1 \cdot y'(x) = 2yy'$$

- **Chain Rule**: $\frac{df(g(x))}{dx} = f'(g(x)) \cdot g'(x)$
- **Specifically**: $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$

- **Inside function**: $g(x) = y(x)$
  - $g'(x) = y'(x) = y'$
- **Outside function**: $f(x) = x^2$
  - $f'(x) = 2x^1$
  - $f'(y(x)) = f'(y(x)) = 2y(x) = 2y$

$$2x + 2yy' = 0 \Rightarrow$$

Solve for $y'$
\[ 2yy' = -2x \implies y' = \frac{-x}{2y} \]

\[ y' = -\frac{y}{2y} \quad \text{(or } \frac{dy}{dx} = -\frac{x}{y}) \]

**Break up \( x^2 + y^2 = 4 \) into two functions and then find derivative of each:**

\[ x^2 + y^2 = 4 \implies y^2 = 4 - x^2 \]

\[ \implies y = \pm \sqrt{4 - x^2} \]

**6.**

\[ y = \sqrt{4 - x^2} \]

\[ y' = \frac{dx}{dx} \left( \sqrt{4 - x^2} \right) \]

\[ = \frac{d}{dx} (4 - x^2)^{1/2} \]

**Chain Rule:**

\[ \frac{dy}{dx} (g(x))^{1/2} = \frac{1}{2} (g(x))^{-1/2} \cdot g'(x) \]

\[ = \frac{1}{2} (4 - x^2)^{-1/2} \cdot (-2x) \]

\[ = \frac{1}{2} (4 - x^2)^{-1/2} \cdot (-2x) \]

\[ = \frac{1}{2} (4 - x^2)^{-1/2} \cdot (-2x) \]
\[ y = -\sqrt{4-x^2} \]

\[ y' = \frac{d}{dx} \left( -\sqrt{4-x^2} \right) \]
\[ = - \frac{d}{dx} \left( 4-x^2 \right)^{\frac{1}{2}} \]
\[ = - \frac{1}{2} \left( 4-x^2 \right)^{-\frac{1}{2}} (2x) \]
\[ = -\frac{x}{\sqrt{4-x^2}} \]

\[ y = -\sqrt{4-x^2} \quad \Rightarrow \quad y = \sqrt{4-x^2} \]
\[ \therefore \quad y' = - \frac{x}{y} \]
HW Exercise 1, p. 245.

Let \( x^2 + 3x + xy = 5 \). Find \( y' \) in 2 ways.

(a) **Implicit Differentiation**:

\[
\frac{d}{dx}(x^2 + 3x + xy) = \frac{d}{dx}(5) \implies \\
\frac{d}{dx}(x^2) + 3 \frac{d}{dx}(x) + \frac{d}{dx}(xy) = \frac{d}{dx}(5) \implies \\
2x + 3 + \left[ (\frac{d}{dx}x) y + x \left( \frac{d}{dx}y \right) \right] = 0 \implies \\
\frac{d}{dx}(xy(x)) = \left[ (\frac{d}{dx}x)y(x) + x \left( \frac{d}{dx}y(x) \right) \right] \\
\text{Product Rule:} \\
\frac{d}{dx}(f(x)g(x)) = (\frac{d}{dx}f(x))g(x) + f(x) \left( \frac{d}{dx}g(x) \right) \\
f(x) = x, \ g(x) = y(x) \\
2x + 3 + \left[ y + x \left( \frac{d}{dx}y \right) \right] = 0 \implies \\
2x + 3 + y + xy' = 0 \implies \\
Solve \text{ for } y'
1) \( xy' = -2x - 3 - y \quad \Rightarrow \)

\[
y' = \frac{-2x - 3 - y}{x} = -2 - \frac{3}{x} - \frac{y}{x}
\]

1) Solve \( x^2 + 3x + xy = 5 \) explicitly for \( y \) and differentiate usual way:

\( x^2 + 3x + xy = 5 \quad \Rightarrow \)

\( xy = 5 - x^2 - 3x \quad \Rightarrow \)

\[
y = \frac{5 - x^2 - 3x}{x} \quad \Rightarrow
\]

\[
y = \frac{5}{x} - \frac{x^2}{x} - \frac{3x}{x} \quad \Rightarrow
\]

\[
y = 5 \cdot \frac{1}{x} - x \cdot \frac{x}{x} - 3 \quad \Rightarrow
\]

\[
y = 5x^{-1} - x - 3
\]

\[
\therefore \quad y' = \frac{d}{dx} \left(5x^{-1} - x - 3\right)
\]

\[
= 5 \cdot (-x^{-2}) - 1 - 0
\]
\[-239\]
\[= -5x^{-2} - 1\]
\[= -5 \cdot \frac{1}{x^2} - 1\]
\[= \frac{-5}{x^2} - 1\]

(c) To see that the solutions in (a) and (b) are the same, substitute
\[y = \frac{5}{x} - x - 3 \quad \text{(from (b))}\]

into
\[y' = \frac{-2x - 3 - y}{x} \quad \text{(from (a))}\]

So,
\[y' = \frac{-2x - 3 - \left(\frac{5}{x} - x - 3\right)}{x}\]
\[= \frac{-2x - 3 - \frac{5}{x} + x + 3}{x}\]
\[ \frac{-x - \frac{5}{x}}{\frac{5}{x}} \]

\[ = \frac{-x}{x} - \frac{\frac{5}{x}}{x} \]

\[ = -1 - \frac{5}{x} \cdot \frac{1}{x} = -1 - \frac{5}{x^2} \]

\[ = -\frac{5}{x^2} - 1 \]
Let \( \sin y + \cos 2y = \cos y \). Find \( \frac{dy}{dx} \) \[ (= y' \]}

\[
\frac{d}{dx}(x \sin y + \cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\frac{d}{dx}(x \sin y) + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\left[ \left( \frac{d}{dx} \right) \sin y + x \left( \frac{d}{dx} \sin y \right) \right] + \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(\cos y) \Rightarrow
\]
\[
\frac{d}{dx}(x \sin y(x)) = \left[ \left( \frac{d}{dx} \right) \sin y(x) + x \left( \frac{d}{dx} \sin y(x) \right) \right]
\]
\[
\text{PRODUCT RULE:}
\]
\[
\frac{d}{dx}(f(x)g(x)) = \left( \frac{d}{dx} f(x) \right) g(x) + f(x) \left( \frac{d}{dx} g(x) \right)
\]
\[
f(x) = x, \quad g(x) = \sin y(x)
\]
\[
\left[ (\cos y + x (\cos y) y') \right] + 2(-\sin 2y) y' = (-\sin y) y' \Rightarrow
\]
\[
\frac{d}{dx}(\sin y(x)) = (\cos y(x)) \cdot y'(x)
\]
\[
\frac{d}{dx}(\cos 2y(x)) = 2(-\sin y(x)) \cdot y'(x)
\]
\[
\text{APPLICATION OF FORMULA}
\]
\[
\frac{d}{dx}(\cos 2y) = 2(-\sin 2y)
\]
\[
\text{AND CHAIN RULE}
\]
\[
\frac{d}{dx}(\cos 2y(x)) = (\cos y(x)) \cdot y'(x)
\]
\[
\text{CHAIN RULE}
\]
\[
\begin{align*}
\sin y + xy'\cos y - 2y'\sin y &= -y'\sin y \\
\text{Solve for } y' \\
x y'\cos y - 2y'\sin y + y'\sin y &= -\sin y \\
y'(x\cos y - 2\sin y + \sin y) &= -\sin y \\
y' &= \frac{-\sin y}{x\cos y - 2\sin y + \sin y}
\end{align*}
\]
4. (HW Exercise 12, p. 245.)

Find an equation to the tangent line to the curve

\[ \frac{x^2}{9} + \frac{y^2}{36} = 1 \]  

[ellipse]

at the point

\[ (x, y) = (-1, 4\sqrt{2}) \]

**Notation:**

slope of tangent line to curve \( \frac{x^2}{9} + \frac{y^2}{36} = 1 \)

at point \((-1, 4\sqrt{2})\) = \( m_{\tan} \)

\[ m_{\tan} = \frac{dy}{dx} \bigg|_{(-1, 4\sqrt{2})} \]

**Note:** \( \frac{x^2}{9} + \frac{y^2}{36} = 1 \) is an implicit equation \( \Rightarrow \) must use implicit differentiation to find \( \frac{dy}{dx} \) and then need to substitute values for \( x \) and \( y \) in the formula for \( \frac{dy}{dx} \).
\[ \frac{d}{dx} \left( \frac{x^2}{9} + \frac{y^2}{36} \right) = \frac{d}{dx} \left( \frac{1}{9} \right) \Rightarrow \]
\[ \frac{d}{dx} \left( \frac{1}{9} \cdot x^2 + \frac{1}{36} \cdot y^2 \right) = \frac{d}{dx} \left( \frac{1}{9} \right) \Rightarrow \]
\[ \frac{1}{9} \frac{d}{dx} (x^2) + \frac{1}{36} \frac{d}{dx} (y^2) = \frac{d}{dx} \left( \frac{1}{9} \right) \Rightarrow \]
\[ \frac{1}{9} (2x) + \frac{1}{36} (2yy') = 0 \Rightarrow \]
\[ \frac{2}{9} x + \frac{1}{18} yy' = 0 \Rightarrow \]
\[ \frac{1}{18} yy' = -\frac{2}{9} x \Rightarrow \]
\[ y' = -\frac{\frac{2}{9} x}{\frac{1}{18} y} \Rightarrow \]
\[ y' = \frac{\frac{3}{9} \cdot x}{\frac{1}{18} y} \Rightarrow \]
\[ y' = \frac{2}{3} \cdot \frac{18}{1} \cdot \frac{x}{y} \Rightarrow \]
\[ y' = \frac{4x}{y} \Rightarrow \]
\[
\frac{dy}{dx} = \frac{4x}{y}
\]
\[d\frac{y}{dx}\bigg|_{(x, y) = (-1, 4\sqrt{2})} = \frac{4(-1)}{4\sqrt{2}} = -\frac{1}{\sqrt{2}}\]

\[m = -\frac{1}{\sqrt{2}}, \quad (x_1, y_1) = (-1, 4\sqrt{2})\]

\[y - y_1 = m(x - x_1) \Rightarrow \]

\[y - 4\sqrt{2} = -\frac{1}{\sqrt{2}}(x + 1)\]

EQ. OF TANGENT LINE
Sections 3.5/3.6/3.7. Remaining Differentiation Formulas for You to Know.

**Inverse Trig Formulas**

Obtained using implicit differentiation and clever tricks in trigonometry.

\[
\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \\
(\text{or} \quad \frac{d}{dx} \arctan x = \frac{1}{1 + x^2})
\]

\[
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1) \\
(\text{or} \quad \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1))
\]
\[
\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)
\]

(or \( \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1) \))

**Examples**

1. (HW Exercise 25, p. 246,)

   Find the derivative of

   \[
   y = \sin^{-1} (x^2)
   \]

   (Use usual differentiation here. Need CHAIN RULE:

   \[
   \frac{d}{dx} \sin^{-1}(f(x)) = \frac{1}{\sqrt{1-(f(x))^2}} \cdot f'(x)
   \]

   )
\[ y' = \frac{d}{dx} \sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx} (x^2) \]
\[ = \frac{2x}{\sqrt{1-x^4}} \]

2. (HW Exercise 29, p. 246.)

Find the derivative of

\[ H(x) = (1+x^2) \arctan x \]

(Use usual differentiation here. Need PRODUCT RULE:)

\[ (f \cdot g)' = f' \cdot g + f \cdot g' \]

\[ H'(x) = \frac{d}{dx} (1+x^2) \arctan x \]
\[ = (\frac{d}{dx} (1+x^2)) \arctan x + (1+x^2) \left( \frac{d}{dx} \arctan x \right) \]
\[ = 2x \arctan x + (1+x^2) \cdot \frac{1}{1+x^2} \]
\[ = 2x \arctan x + 1 \]
Exponential Formula

Obtained using the chain rule and clever tricks in algebra.

\[ \frac{d}{dx} a^x = a^x \ln a \]

Note: When \( a = e \), we have

\[ \frac{d}{dx} e^x = e^x \ln e = e^x \cdot 1 = e^x \]

Example. Find the derivative of

\[ F(x) = 3x^2 + 5x + 1 \]

(Use usual differentiation here, need CHAIN RULE:

\[ \frac{d}{dx} a^{f(x)} = a^{f(x)} \ln a \cdot f'(x) \] )
\[ F'(x) = \frac{d}{dx} \left( 3x^3 + 5x + 1 \right) \]
\[ = 9x^2 + 5 + \frac{d}{dx}(\ln 3 \cdot (3x^2 + 5x + 1)) \]
\[ = 9x^2 + 5 + (\ln 3)(2x + 5) \]
Log Formulas

Obtained using implicit differentiation and the definition of log:

\[ \log_a x = y \text{ means } a^y = x \]

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

\[ \frac{d}{dx} \log_a x = \frac{1}{x \ln a} \]

Note: When \( a = e \), we have \( \log_e x = \ln x \)

\[ \frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{x \ln e} = \frac{1}{x \cdot 1} = \frac{1}{x} \]
Examples.

1. (Exercise 2, p. 252.)

Find the derivative of

$$f(x) = \ln(2-x)$$

(Use usual differentiation here. Need CHAIN RULE:

$$\frac{d}{dx} \ln g(x) = \frac{1}{g(x)} \cdot g'(x)$$)

$$f'(x) = \frac{d}{dx} \ln(2-x)$$

$$= \frac{1}{2-x} \cdot \frac{d}{dx} (2-x)$$

$$= \frac{1}{2-x} \cdot (0-1)$$

$$f'(x) = \frac{-1}{2-x}$$

$$f'(x) = \frac{1}{x-2}$$
2. (Exercise 4, p. 252)

Find the derivative of

\[ f(x) = \cos(\ln x) \]

(Use usual differentiation here. Need CHAIN RULE:)

\[ \frac{d}{dx} \cos(g(x)) = -\sin(g(x)) \cdot g'(x) \]

\[ f'(x) = \frac{d}{dx} \cos(\ln x) \]

\[ = -\sin(\ln x) \cdot \frac{1}{x} \]

\[ = \frac{-\sin(\ln x)}{x} \]

 ALSO try on your own Exercises 3, 5, 9, 10, 17 on p. 252 of Sect. 3.7
CHAPTER 4 Applications of Differentiation.

We will first apply IMPPLICIT DIFFERENTIATION to find rates of change of processes such as the rate of increase of volume of an expanding balloon.

Then we will use the FIRST AND SECOND DERIVATIVES of a function to maximize or minimize quantities such as the cost of making a product! Involved here is the following idea:

\[ f'(a) = 0 \Rightarrow \text{At } x = a \text{ we have a "peak" (max), "valley" (min) or "slope" (neither) in the graph of } y = f(x) \]

![Graphs showing peak, valley, and slope](image-url)
We might be interested in the following rates of change:

\[
\frac{dr}{dt} = \text{rate of increase of radius per unit increase in time}
\]

\[
\frac{dA}{dt} = \text{rate of increase of area per unit increase in time}
\]

\[
\frac{dA}{dr} = \text{rate of increase of area per unit increase in radius}
\]

\[
\frac{dC}{dt} = \text{rate of circumference per unit increase in radius}
\]

e.t.c.,

In the rate of change problems we will consider, we will mainly want to find the rate of change of a quantity \( Q \) per unit time by knowing

\[
\text{time rates of change of quantities} \, \text{so} , \, r \, \text{and} \, \ell \, \text{that are related to} \, Q
\]
Section 4.1. Related Rates.

Recall: The derivative \( \frac{dy}{dx} \) gives instantaneous rate of change of \( y = f(x) \) with respect to \( x \), where

\[
\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \\
= \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x}
\]

A rate change in \( y \) with respect to change in \( x \).

Example. Suppose a pebble is dropped into a calm pond and a circular wave spreads out like so:

PEBBLE DROPPED

POND
Outline for Related-Rate Problems

(SEE HANDOUT)

Suggestions: (1) Read problem carefully,
(2) Draw figure where appropriate and label
(3) Then follow the steps below.

STEP 1. Decide what rate of change is desired and express it in \( \frac{dy}{dx} \) notation:

Find \( \frac{dQ}{dt} \) when \( \ldots \).

STEP 2. Decide what rates of change are given and express these data in \( \frac{dy}{dx} \) notation:

Given \( \frac{dQ}{dt} = \ldots \) and \( \frac{dC}{dt} = \ldots \)

when \( \ldots \).

STEP 3. Find an equation relating \( Q, r, \) and \( s \).
You may have to use a figure or some geometric formula.
STEP 4. Differentiate, with respect to time \( t \), the relation equation in STEP 3 (often by IMPLICIT DIFFERENTIATION) to obtain a relation among \( \frac{dq}{dt} \), \( \frac{dr}{dt} \), \( \frac{dc}{dt} \).

STEP 5. Put in values of \( r, s, Q \) and of \( \frac{dr}{dt}, \frac{dc}{dt} \) corresponding to the instant when \( \frac{dq}{dt} \) is desired, and solve for \( \frac{dq}{dt} \).
Examples.

1. If the radius \( r \) of a circular disk is increasing at the rate of 2 inches per second, find the rate of increase of its area when \( r = 4 \) inches.

\[
A = A(t) = \text{area as a function of time} \\
r = r(t) = \text{radius as a function of time}
\]

**STEP 1.** Find \( \frac{dA}{dt} \) when \( r = 4 \) in.

**STEP 2.** Given \( \frac{dr}{dt} = 3 \text{ in/sec} \) when \( r = 4 \) in.

**STEP 3.** Formula relating \( A \) and \( r \) of a circle:

\[
A = \pi r^2
\]

**STEP 4.** Implicit Differentiation of \( A = \pi r^2 \) with respect to \( t \) where \( A = A(t) \) and \( r = r(t) \).
\[ \frac{d}{dt} (A) = \frac{d}{dt} \left( \pi r^2 \right) \Rightarrow \]
\[ \frac{d}{dt} (A) = \pi \frac{d}{dt} (r^2) \Rightarrow \]
\[ \frac{dA}{dt} = \pi (2r \frac{dr}{dt}) \Rightarrow \]
\[ \boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}} \]

Just like implicitly differentiating \( x^2 + y^2 = 1 \) (where \( t \) is like \( x \) and \( A \) and \( r \) are like \( y \)): \[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1) \Rightarrow \]
\[ \frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) = \frac{d}{dx} (1) \Rightarrow \]
\[ 2x + 2y \frac{dy}{dx} = 0 \]

**STEP 5.** When \( r = 4 \), we have

\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi (4)(3) = \boxed{24 \text{ in}^2 \text{ sec}} \]

If a snowball melts so that its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 10 cm.

\[ r = r(t) = \text{radius} \]
\[ D = D(t) = \text{diameter} \]
\[ A = A(t) = \text{surface area} \]

**STEP 1.** Find \( \frac{dA}{dt} \) when \( D = 10 \text{ cm} \).

**STEP 2.** Given \( \frac{dA}{dt} = -1 \text{ cm}^2/\text{min} \) when \( D = 10 \text{ cm} \).

**IMPORTANT:**

E.g., \( \frac{dA}{dt} = -1 \): Surface area decreases at a rate of 1 cm²/min.

E.g., \( \frac{dA}{dt} = 1 \): Surface area increases at a rate of 1 cm²/min.
**STEP 3. Formula relating $A$ and $D$ of a sphere:**

From front cover of text under "sphere" ⇒

$$A = 4\pi r^2 \Rightarrow \quad D = 2\pi \Rightarrow r = \frac{D}{2}$$

$$A = 4\pi \left(\frac{D}{2}\right)^2 \Rightarrow$$

$$A = \pi \frac{D^2}{4} \Rightarrow$$

$$A = \pi D^2$$
STEP 5. When $D = 10$, we have

\[ \frac{dA}{dt} = 2\pi D \frac{dD}{dt} \]

Want to solve for $\frac{dD}{dt}$
WHERE KNOW  $\frac{dA}{dt} = -1$ and $D = 10$

\[ \frac{dD}{dt} = \frac{1}{2\pi B}, \quad \frac{dA}{dt} = \frac{1}{2\pi (11)} \cdot (-1) = \frac{1}{2\pi} \frac{\text{cm}}{\text{min}} \]

\[ \frac{1}{9/10} \]

\[ \text{Diameter is decreasing at a rate of } \frac{1}{20\pi} \frac{\text{cm}}{\text{min}}. \]
Lecture

Sections 4.2 and 4.3.

Maximum/Minimum Values of Functions and Derivatives/Shapes of Curves.

Here we will determine what values of $x$ will give us the largest and smallest (absolutely or relatively) values of $f(x)$.

Very often this kind of information is needed in applications.

Given a function $f(x)$, we will be making use of $f'(x)$ and $f''(x)$.

We will be interested in points $x$ where the graph of $f(x)$ changes in some way. Specifically:

1) peaks: $f_{\text{inc}} \Rightarrow f_{\text{dec}}$

2) valleys: $f_{\text{dec}} \Rightarrow f_{\text{inc}}$

3) critical points: includes where peaks and valleys occur
4) Inflection points: where concavity (bending of a curve) changes

1. Local Maximum: • a peak (among other peaks)

Locally, this is the highest point around.

- with derivative $f' = 0$
- $f''(p) = 0$
- $f'(x) > 0$ (inc.)
- $f'(x) < 0$ (dec.)

Concave down with $f''(c) < 0$
2. **Local Minimum**: a valley (among other valleys)

   ![Graph showing a local minimum]

   Locally, this is the lowest point around.

   - concave up with $f''(p) > 0$
   - $f$ inc. $\Rightarrow f'(q) > 0$
   - $f$ dec. $\Rightarrow f'(q) < 0$

   $f(p) = 0$

3. **Critical Points**

   - Define. $x = p$ is a critical point if either

     $$f'(p) = 0$$

   or

     $$f'(p) = \text{undefined } (=\infty) .$$

   ![Graph showing critical points]

   - $f'(x)$ undefined $\Rightarrow x = 0$ is a critical point.
• **WARNING**: If \( f \) has a local max or min at \( x=p \), then \( f'(p)=0 \).

However, if \( f'(p)=0 \), \( f \) may or may not have a local max or min at \( x=p \).

\[
E.g., \quad y = x^2
\]

\( f'(x) = 0 \) but there is no local max or min at \( x=0 \).

---

First/Second Derivative Tests for Locating Local Maximum and Minimum

- **Local max**
  \[
  f(p)=0, \quad f''(p)<0
  \]
  \[
  f'(x)<0, \quad f''(p)<0
  \]
  \[
  f'(x)>0, \quad f''(p)>0
  \]
  \[
  f''(p)>0
  \]
  \[
  f'(x)<0
  \]

- **Local min**
  \[
  f(p)=0, \quad f''(p)>0
  \]
  \[
  f'(x)>0, \quad f''(p)>0
  \]
  \[
  f'(x)<0
  \]
  \[
  f''(p)>0
  \]
  \[
  f'(x)>0
  \]
**FIRST DERIVATIVE TEST:**

Assume $f'(p) = 0$.

1. If $f'(x) > 0$ "right before" $x = p$ and $f'(x) < 0$ "right after" $x = p$, then $f(x)$ has a LOCAL MAX at $x = p$.

2. If $f'(x) < 0$ "right before" $x = p$ and $f'(x) > 0$ "right after" $x = p$, then $f(x)$ has a LOCAL MIN at $x = p$.

3. OTHERWISE, $f'(x)$ DOES NOT HAVE A LOCAL MAX or MIN at $x = p$.

---

**SECOND DERIVATIVE TEST**

Assume $f''(p) = 0$.

1. If $f''(p) < 0$, then $f(x)$ has a LOCAL MAX at $x = p$.

2. If $f''(p) > 0$, then $f(x)$ has a LOCAL MIN at $x = p$.

3. If $f''(p) = 0$, then this TEST FAILS. GO TO FIRST DERIVATIVE TEST.
Example. \( f(x) = x^3 \):

\[
\frac{df}{dx}(x^3) = 3x^2
\]

Critical Point

\[
\frac{d}{dx}(3x^2) = 6x
\]

\[
f''(0) = 6(0) = 0 \quad \text{TEST FAILS}
\]

Try 1st Deriv. Test

\[
1^{\text{st}} \text{ Deriv. Test:}
\]
The function $f(x) = x^2$ is given. To find its critical points, we first compute its derivative.

$$f'(x) = \frac{d}{dx} (x^2) = 2x$$

Next, we set $f'(x) = 0$ to find the critical points:

$$2x = 0 \Rightarrow x = 0$$

Thus, $x = 0$ is a critical point of $f(x) = x^2$. We can classify this point by checking the sign of the derivative on either side of $x = 0$.

For $x < 0$, choose a sample point $x = -1$. The derivative is $f'(-1) = 3(-1)^2 = 3 > 0$, indicating that $f(x)$ is increasing right before $x = 0$.

For $x > 0$, choose a sample point $x = 1$. The derivative is $f'(1) = 3(1)^2 = 3 > 0$, indicating that $f(x)$ is increasing right after $x = 0$.

Since $f(x)$ is increasing before and after $x = 0$, $x = 0$ is neither a local maximum nor a local minimum. It is simply a critical point.
Second Derivative Test:

\[ f''(x) = \frac{d}{dx}(2x) = 2 \]

\[ f''(0) = 2 > 0 \quad \Rightarrow \quad f(x) = x^2 \text{ has a local min at } x = 0 \]

plug in critical point

---

Concavity and the Test for an Inflection Point:

The following is a definition of inflection point, which also serves as a test for an inflection point.

\[ x = p \text{ is an inflection point of } f(x) \text{ if} \]

1. \( f''(p) = 0 \)
2. concavity changes at \( x = p \) where either
   a. \( f''(x) > 0 \) "right before" \( x = p \) and \( f''(x) < 0 \) "right after" \( x = p \)
   or b. \( f''(x) < 0 \) "right before" \( x = p \) and \( f''(x) > 0 \) "right after" \( x = p \)

\[ \text{MEMORIZE} \]
Example. \( f(x) = x^3 \). 

\[
\begin{align*}
 f'(x) &= \frac{d}{dx} (x^3) = 3x^2 \\
 f''(x) &= \frac{d}{dx} (3x^2) = 6x
\end{align*}
\]

\( f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow (x = 0) \)

**Candidate for Inflection Point**

\[
\begin{array}{c|c|c|c}
 f''(-1) = ? & f''(0) = 0 & f''(1) = ? \\
\hline
-1 & 0 & 1
\end{array}
\]

These are sample points chosen "right before" and "right after" \( x = 0 \)

\[
\begin{align*}
 f''(-1) &= 6(-1) = -6 < 0 \Rightarrow f''(x) < 0 \text{ right before } x = 0 \\
 f''(1) &= 6(1) = 6 > 0 \Rightarrow f''(x) > 0 \text{ right after } x = 0
\end{align*}
\]

\( x = 0 \) is an **Inflection Point** of \( f(x) \)
Example. Inflexion points are always between local max and min:

\[ x = p \text{ is an inflexion point} \]
Examples.

1. (HW Exercise 2, p. 292, Sect. 4.3)

(a) Where is \( g \) concave up? (Give open intervals.)
(b) Where is \( g \) concave down? (Give open intervals.)
(c) What are the inflection points of \( g \) ?

(a) \([-2, 2) \cup (7, 9]\)  
(b) \((2, 7) \cup (4, 7]\)
(c) \[ x = 2 \]

\[ x = 2 \quad \text{or} \quad (x, y) = (0, 0) \]

is not a point of inflection, even though \( y \) changes concavity from concave down to concave up.
2. (HW Exercise 5, p. 292, Sect. 4.3.)

What are the inflection points of \( f \)?

Graph of \( y = f''(x) \), not \( y = f(x) \! \)

Inflection pts are \((0+0^+)\) \(x = 1\) and \(x = 7\) where \( f'' \) changes sign (from neg. to pos., or pos. to neg.,)

(\( x = 4 \) is \( 0^- \) an inflection point, since \( f'' \) does not change sign !).
Let \[ f(x) = x^4 + 12x^2 + 16x^2. \]

Find all local maxima and minima.

\[ f(x) = x^4 + 12x^2 + 16x^2 \]
\[ f'(x) = 4x^3 + 36x^2 + 32x \]
\[ f''(x) = 12x^2 + 72x + 32 \]

Set \( f'(x) = 0 \) and solve for \( x \):
\[ f'(x) = 0 \Rightarrow 4x^2 + 36x + 32x = 0 \]
\[ \Rightarrow 4x(x^2 + 9x + 8) = 0 \]
\[ \Rightarrow 4x(x+8)(x+1) = 0 \]
\[ \Rightarrow x = 0, x = -1, x = -8 \]

3 CRITICAL POINTS
(candidates for being local max's and min's)
2nd Deriv. Test:

\[ f''(0) = 12(0)^2 + 72(0) + 32 = 32 > 0 \implies f(x) \text{ has a local min at } x = 0 \]

\[ f''(-1) = 12(-1)^2 + 72(-1) + 32 = -28 < 0 \implies f(x) \text{ has a local max at } x = -1 \]

\[ f''(-8) = 12(-8)^2 + 72(-8) + 32 = 224 > 0 \implies f(x) \text{ has a local min at } x = -8 \]

Sketch of \( y = x^3 + 12x^2 + 16x \):
3. (HW Exercise 4, p. 292, Sect. 4.3.)

Let

\[ f(x) = x^6 + 192x + 17 \]

(a) Where is \( f \) increasing and decreasing?

(b) Where are the local max and min of \( f \)?

(c) Where is \( f \) concave up and down? What are the inflection points of \( f \)?

\[ f'(x) = 6x^5 + 192 \]

\[ f''(x) = 30x^4 \]

Set \( f'(x) = 0 \) \( \Rightarrow \)

\[ 6x^5 + 192 = 0 \]

\[ 6x^5 = -192 \]

\[ x^5 = -\frac{192}{6} \]

\[ x = -2 \] (CRITICAL POINT)

(critical for being a local max or min)
Check to see if $f$ has a local max or min using the FIRST DERIVATIVE TEST:

$$f' < 0 \quad f' = 0 \quad f' > 0$$

Sample points "right before" and "right after" $x = -2$

$$f'(-3) = 6(-3)^5 + 192 = 6(-243) + 192 = -1458 + 192 = -1266 < 0$$

$$f'(-1) = 6(-1)^5 + 192 = 6(-1) + 192 = -6 + 192 = 186 > 0$$

$$f' < 0 \quad f' = 0 \quad f' > 0$$

LOCAL MIN
The local min value of \( f \) is then

\[
f(-2) = (-2)^6 + 192(-2) + 17 = 64 - 384 + 17
\]

plug in \( p = -\frac{303}{303} \)

at which local min occurs

Check to see if \( f \) has any inflection points:

Set \( f''(x) = 0 \) \( \Rightarrow 30x^4 = 0 \)

\( x = 0 \)

Candidate for being an inflection point

\[
\begin{array}{c|c|c}
f'' & f'' = 0 & f'' \geq 0 \\
-1 & 0 & 1 \end{array}
\]
\[ f''(-1) = 30 \times (-1)^4 = 30 \Rightarrow 0 \]
\[ f''(1) = 30 \times (1)^4 = 30 \Rightarrow 0 \]

\( x = 0 \) is NOT an inflection point of \( f \), since \( f'' \) does not change sign around \( x = 0 \)!

**Sketch of \( f(x) \):**

**Graph is always increasing after \( x = -2 \), since no local max to change that, like so:**

\( x = -2 \)
SUMMARY:

- $f$ has one local min at $x = -2$.
- $f$ has no local max.
- $f$ has no inflection points.
- $f$ is decreasing on $(-\infty, -2)$.
- $f$ is increasing on $(-2, \infty)$.
- $f$ is concave up on $(-\infty, \infty)$. 
Finding Global (or Absolute) Maxima and Minima

We now will focus primarily on Section 4.2. Previously we were focusing on Section 4.3.

We will consider

**Continuous Functions**

(i.e., no holes or breaks or jumps in their graphs) on

**Closed Intervals**

(i.e., \(a \leq x \leq b\) or \([a, b]\) versus \(a < x < b\) or \((a, b)\)).
We will look for the following:

1. THE **global** (or absolute) maximum value of \( f \), which may be achieved at one or more locations in the graph of \( y = f(x) \) on \([a, b]\).

2. THE **global** (or absolute) minimum value of \( f \), which may be achieved at one or more locations in the graph of \( y = f(x) \) on \([a, b]\).

We are given a guarantee by the following theorem:

**THEOREM.** Let \( f \) be a continuous function on the closed interval \([a, b]\). Then \( f \) will have exactly one global maximum value and exactly one global minimum value.

**Remark.** \( f \) may have many or no local maximum values or local minimum values.

**RULE OF THUMB.**

1. **Global maximum value of** \( f \) will occur at one (or more) of the local maxima of \( f \) or at one (or both) of the endpoints, \( x = a \) and \( x = b \).
2. Global minimum value of $f$ will occur at one (or more) of the local minima of $f$ or at one (or both) of the endpoints, $x=a$ and $x=b$.

E.g.,

- In the first graph, there is a local max and a global max, and the global min is at one of the endpoints.
- In the second graph, there are multiple local maxima and minima, and the global max occurs at an internal point, while the global min occurs at one of the endpoints.
- In the third graph, there is a local max and a global max, and the global min occurs at one of the endpoints.
We are given a recipe to find locations of global max and min of \( f \) when \( f \) is continuous on a closed interval \([a, b]\).

**RECIPE**

**STEP 1.** Find all critical points of \( f(x) \);

where \( f'(x) = 0 \) (or \( f'(x) = \infty \)).

Optional: Determine if critical pts correspond to local max and/or min.

**STEP 2.** Evaluate \( f(x) \) at EVERY critical point AND at BOTH endpoints \((x = a, x = b)\).

**STEP 3.** Global max value = largest value of \( f(x) \) from **STEP 2**

Global min value = smallest value of \( f(x) \) from **STEP 2**
Example. (HW Exercise 38, p. 280.)

Find absolute (= global) max/min values of

\[ f(x) = x^3 - 12x + 1 \text{ on } [-3, 5] \]

Note: \( f(x) = \text{polynomial} \Rightarrow f(x) \text{ is continuous on } (-\infty, \infty) \Rightarrow f(x) \text{ is certainly continuous on } [-3, 5] \).

**STEP 1. Critical Points:**

\[ f'(x) = 3x^2 - 12 \]

\[ f'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \]

\[ 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2 \]

**STEP 2. Evaluate \( f(x) \) at \( x = -2, 2, -3, 5 \):**

- \( f(-2) = (-2)^3 - 12(-2) + 1 = 17 \)
- \( f(2) = (2)^3 - 12(2) + 1 = -15 \) \( \Rightarrow \text{SMALLEST} \)
- \( f(-3) = (-3)^3 - 12(-3) + 1 = 10 \)
- \( f(5) = (5)^3 - 12(5) + 1 = 66 \) \( \Rightarrow \text{LARGEST} \)
STEP 3. Absolute (= global) max/min values of \( f(x) \):

\[
\begin{align*}
\text{f(2)} &= -15 \quad \text{absolute min value at } x = 2 \\
\text{f(5)} &= 66 \quad \text{absolute max value at } x = 5
\end{align*}
\]
Section 4.8. Antiderivatives

The Family of Antiderivatives Associated With a Function $f$

Antiderivative = opposite of derivative

Definition. $F$ is an antiderivative of $f$ if its derivative is equal to $f$, i.e.,

$$F'(x) = f(x).$$

Example. The function $f(x) = 3x^2$ is the derivative of another function $F(x)$, called the antiderivative of $f(x)$. Find (or guess at) $F(x)$.

$$F'(x) = f(x) \Rightarrow F'(x) = 3x^2$$

$$\Rightarrow F(x) = x^3$$

Check: $F'(x) = \frac{d}{dx}(x^3) = 3x^2 = f(x) \checkmark$
\[
\begin{align*}
\text{differentiation} & \quad \begin{cases} 
3x^2 \\
3x^2 
\end{cases} \\
\text{antidifferentiation} & \quad \begin{cases} 
x^3 \\
x^3 
\end{cases}
\end{align*}
\]
\[
\text{antiderivative} \quad \text{derivative}
\]

Actually, other \( F(x) \)'s that will also work are:

\[
\begin{align*}
F(x) &= x^3 + 1 \quad ; \quad F'(x) = \frac{d}{dx} (x^3 + 1) = 3x^2 = f(x) \checkmark \\
F(x) &= x^3 - 55 \quad ; \quad F'(x) = \frac{d}{dx} (x^3 - 55) = 3x^2 = f(x) \checkmark \\
F(x) &= x^3 + \frac{3}{4} \quad ; \quad F'(x) = \frac{d}{dx} (x^3 + \frac{3}{4}) = 3x^2 = f(x) \checkmark
\end{align*}
\]

All of these \( F(x) \)'s are antiderivatives of \( f(x) = 3x^2 \).

So, we say that the general antiderivative or family of antiderivatives of \( f(x) = 3x^2 \) is

\[
F(x) = x^3 + C \quad ; \quad C = \text{any constant}
\]
Check: \( F'(x) = \frac{d}{dx} (x^3 + C) = 3x^2 + 0 = 3x^2 = f(x) \) \( \checkmark \)
Visualizing Antiderivatives Using the Slopes of Tangent Lines

(SEE Section 2.10, pp. 199-198, as well as Section 4.9, p. 332)

Recall that

\[ F'(x) = \text{slope of tangent line to } \ y = F(x) \text{ at } x = a \]

Derivative = Slope

We will first use this idea in an attempt to sketch \( F(x) \) given a sketch of \( F'(x) = f(x) \), where \( F(x) \) is the antiderivative of \( f(x) \):

\[ F'(x) \quad (= f(x)) \]

\[ \downarrow \]

\[ x \]

\[ F(x) ? \]

\[ \downarrow \]

\[ x \]
Example. (Exercise 2, p. 174, Sect. 2.10)

The graph of the derivative \( F' \) of a function \( f \) is shown:

(a) On what intervals is \( F \) increasing or decreasing?
(b) At what values of \( x \) does \( F \) have a local maximum or minimum?
(c) If it is known that \( F(a) = 0 \), sketch a possible graph of \( F \).
\[ y = F'(x) \]

\[ y = F(x) \]

\[
\begin{array}{c|c}
F' & F \\
\hline
> 0 & \text{increasing} \\
> 0 & \text{local max} \ 	ext{or local min} \\
\leq 0 & \text{decreasing} \\
\leq 0 & \text{concave up} \\
> 0 \text{ or } < 0 & \text{concave down} \\
\end{array}
\]
We will now sketch $F(x)$ by connecting a whole bunch of short tangent lines that come from $F'(x) = f(x)$.

$\begin{align*}
F(x) &= 2x^2 + 2, \quad F(0) = 2 \\
F(x) &= 3x^2 + 1, \quad F(0) = 1 \\
F'(x) &= 3x \\
F'(0) &= 0 \\
F'(2) &= 6
\end{align*}$

Consider a direction field for $f(x) = 3x^2$.
Example (Exercise 32, p. 334, Sect. 4.9.)

Use a direction field to graph the antiderivative of

\[ f(x) = x \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \]

that satisfies

\[ F(0) = 0. \]

\[ f(x) = x \tan x = \text{slope of tangent lines at } x \]
\[ f(x) = \tan x = \text{slope of tangent lines at } x \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>±( \pi/8 )</td>
<td>0.16</td>
</tr>
<tr>
<td>±( \pi/4 )</td>
<td>0.78</td>
</tr>
<tr>
<td>±3( \pi/8 )</td>
<td>2.84</td>
</tr>
<tr>
<td>±( \pi/2 )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
The ANTIDERIVATIVE of a function $f$ is also called the (INDEFINITE) INTEGRAL of a function $f$.

The indefinite integral of a function $f$ is denoted by

$$ \int f(x) \, dx = F(x) + C $$

General antiderivative (or family of antiderivatives) of $f(x)$ where

$$ \frac{d}{dx} (F(x) + C) = f(x) + 0 $$

$$ = F'(x) $$

$$ = f(x) $$
So we have

**DIFFERENTIATION**, \( \frac{d}{dx} \)

and

**INTEGRATION**, \( \int dx \)

Two reverse processes:

**DIFFERENTIATION**

\[
\frac{d}{dx} (F(x) + C) = F'(x) = f(x)
\]

**ANTIDERIVATIVE** \( F(x) + C \)

**DERIVATIVE** \( f(x) \)

\[ \int f(x) \, dx = F(x) + C \]

**INTEGRATION**
Examples.

1. Antiderivative of \( f(x) = 0 \)?
   - Ask: What function has derivative = 0?
   - Answer: Any constant function \( F(x) = C \).
   - So
   \[
   \int 0 \, dx = C
   \]

2. Antiderivative of \( f(x) = 3 \)?
   - Ask: What function has derivative = 3?
   - Answer: The linear function \( F(x) = 3x \).
   - So
   \[
   \int 3 \, dx = 3x + C
   \]
   - "Tag on" a \( C \).

3. Antiderivative of \( f(x) = x \)?
   - Ask: What function has derivative = \( x \)?
   - Answer: Try \( F(x) = x^2 \). Then
   \[
   F'(x) = 2x \quad \Rightarrow \quad x = f(x)
   \]
   - Try \( F(x) = \frac{1}{2} x^2 \). Then
   \[
   F'(x) = x = f(x) \quad \checkmark
   \]
\[ \int x \, dx = \frac{1}{2} x^2 + C \]

"Tag on" a C.

4. Antiderivative of \( f(x) = x^2 \)?

Ask: What function has derivative \( x^2 \)?

Answer: Try \( F(x) = x^3 \). Then

\[ F'(x) = 3x^2 \neq x^2 = f(x) \text{ almost} \]

Try \( F(x) = \frac{1}{3} x^3 \). Then

\[ F'(x) = x^2 = f(x) \checkmark \]

So

\[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

"Tag on" a C.
In general:

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \]

or \( \frac{1}{n+1}, x^{n+1} \)

Otherwise, can get

\[ \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \text{undefined!} \]
More Examples.

1. Antiderivative of $f(x) = e^x$?
What function has derivative $e^x$?
Recall: $\frac{d}{dx}(e^x) = e^x$.

So

$$\int e^x \, dx = e^x + C$$

(derivative of this must equal this)

2. Antiderivatives of $f(x) = \sin x$ and $f(x) = \cos x$?
What function has derivative $\sin x$? $\cos x$?
Recall: $\frac{d}{dx}(\cos x) = -\sin x$ \Rightarrow

$$\frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(-\sin x) = \sin x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

So

$$\int \sin x \, dx = -\cos x + C$$
$$\int \cos x \, dx = \sin x + C$$
3. Antiderivative of \( f(x) = x^{-1} = \frac{1}{x} \)?

**NOTE:** Cannot use formula
\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.
\]

What function has derivative \( \frac{1}{x} \)?

Actually, there are two!

\( x > 0 \):
\[
F(x) = \ln(x) \quad \text{positive}
\]
\[
\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln(x)) = \frac{1}{x}
\]

\( x < 0 \):
\[
F(x) = \ln(-x) \quad \text{positive}
\]
\[
\frac{d}{dx}(F(x)) = \frac{d}{dx}(\ln(-x)) = \frac{1}{x} \cdot \frac{d}{dx}(-x) = \frac{1}{x} \cdot (-1) = \frac{-1}{x}
\]

**Chain Rule**
\[
\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)
\]

Can put these two \( F(x) \)'s together into one function

\[
F(x) = \begin{cases} 
\ln(x), & x > 0 \\
\ln(-x), & x < 0
\end{cases}
\]

So
\[
\int \frac{1}{x} \, dx = \ln|x| + C
\]
Properties of Antiderivatives
(or Indefinite Integrals)

1. \( \int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)

   \( \int (x^2 + \frac{1}{x}) \, dx = \int x^2 \, dx + \int \frac{1}{x} \, dx \)
   \[ = \frac{x^3}{3} + \ln|x| + C \]

   Just "tag on" \( C \)

2. \( \int c f(x) \, dx = c \int f(x) \, dx \)

   \( \int 3e^x \, dx = 3 \int e^x \, dx \)
   \[ = 3e^x + C \]

   "Tag on" \( C \) lost
Examples. Find the most general antiderivative \( F(x) + C \) of the function \( f(x) \).

1. (Exercise 2, p. 334)

\[
\int f(x) \, dx = \int (1 - x^2 + 12x^5) \, dx
\]

\[
= \int x^0 \, dx - \int x^2 \, dx + 12 \int x^5 \, dx
\]

\[
= \frac{x^{0+1}}{0+1} - \frac{x^{3+1}}{3+1} + 12 \cdot \frac{x^{5+1}}{5+1} + C
\]

\[
= \frac{x^1}{1} - \frac{x^4}{4} + \frac{12 \cdot x^6}{6} + C
\]

\[
= x - \frac{x^4}{4} + 2 \cdot x^6 + C
\]

Check:

\[
\frac{d}{dx} \left( x - \frac{x^4}{4} + 2 \cdot x^6 + C \right)
\]

\[
= 1 - \frac{4x^3}{4} + 12 \cdot x^5 + 0
\]

\[
= 1 - x^2 + 12x^5 \checkmark
\]
3. (Exercise 10, p. 334.)

\[ f(x) = 3e^x + 7 \sec^2 x \]

\[
\int f(x) \, dx = \int (3e^x + 7\sec^2 x) \, dx
= 3 \int e^x \, dx + 7 \int \sec^2 x \, dx
= 3e^x + 7 \tan x + C
\]

Recall: \( \frac{d}{dx}(\tan x) = \sec^2 x \)
4. (Exercise 12, p. 334.)

\[ f(x) = \frac{x^2 + x + \frac{1}{x}}{x} \]

\[ = \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x} = x + 1 + \frac{1}{x} \]

\[ \int f(x) \, dx = \int \left( x + 1 + \frac{1}{x} \right) \, dx \]

\[ = \int x \, dx + \int 1 \, dx + \int \frac{1}{x} \, dx \]

\[ = \frac{x^2}{2} + x + \ln|x| + C \]
Example. (Exercise 14, p. 334.)

Find antiderivative $F$ of $f$ that satisfies given condition.

$$f(x) = 4 - 3(1 + x^2)^{-1} = 4 - 3 \cdot \frac{1}{1 + x^2}$$

$F(0) = 4$

Called "given condition" or "initial condition." We will use this to determine a specific value for $C$.

$$\int f(x) \, dx = \int \left( 4 - 3 \cdot \frac{1}{1 + x^2} \right) \, dx$$

$$= 4 \int 1 \, dx - 3 \int \frac{1}{1 + x^2} \, dx$$

$$= 4x - 3 \arctan x + C = F(x)$$

Recall: $\frac{d}{dx} (\arctan x) = \frac{1}{1 + x^2}$

Apply $F(0) = 4$ to $F(x) = 4x - 3\arctan x + C$

to determine $C$.
\[ 2 \tan^{-1}(31) \]

\[ 4 = F(0) = 4(0) - 3 \tan^{-1}(0) + C \quad \Rightarrow \]
\[ 4 = 4(0) - 3 \tan^{-1}(0) + C \quad \Rightarrow \]
\[ = 0 \text{ by calculator} \]
\[ 4 = 0 - 3(0) + C \quad \Rightarrow \]
\[ 4 = C \quad \Rightarrow \]
\[ C = 4 \]

\[ F(x) = 4x - 3 \tan^{-1}x + 4 \]
CHAPTER 5  Integrals.

Ch. 2  vs.  Ch. 5

We computed velocity, \( v \), from distance traveled, \( s \); traveled, \( s \), from velocity, \( v \).

\[ v(t) = s'(t) \text{ or } \frac{ds}{dt} \]

This led us to the idea of the derivative for functions in general.

We will compute distance traveled, \( s \), from velocity, \( v \),

This will lead us to the definite integral.

Inverse processes:

Differentiation (= finding the derivative)
Integration (= finding the integral)

E.g., let \( f(x) = x^2 \). Then \( \frac{d}{dx} x^2 = 2x \).

Differentiation and integration are related in the following way:
CHAPTER 5. Integrals.

You were just introduced to the 

\[ \int f(x) \, dx = F(x) + C = \text{expression in } x \]

\[ \frac{d}{dx} F(x) = f(x) \]

In this chapter you will be introduced to the "other" integral, the 

\[ \int_{a}^{b} f(x) \, dx = \text{a number} \]

where numbers a and b are plugged into F(x),
DIFFERENCEIATION

\[ \frac{d}{dx} x^2 = 2x \]

\[ \int 2x \, dx = x^2 \]

INTEGRATION
We will SKIP Sect. 5.1 and most likely SKIP Sects. 5.2 and 5.4 (so we will go directly to 5.3 and then 5.5).
Sects. 5.1, 5.2, and 5.4 show the following:

*First Recap:*

Sect. 2.1 & Ch. 3. The derivative of a function $f(x)$ is itself a function or expression in $x$, e.g.,

$$f(x) = x^2 \Rightarrow f'(x) = \frac{d}{dx} (x^2) = 2x$$

$$\Rightarrow f'(x) = 2x$$

Sect. 2.6 & 2.8. The derivative of a function $f(x)$ at a point $x = a$ is

$$f'(a) = \text{a number}$$

which is the slope of the tangent line to the curve $y = f(x)$, at $x = a$

* e.g., $f(x) = x^2$, $a = 1$

$$f'(x) = 2x \Rightarrow f'(1) = 2(1) = 2$$
\[ \int_0^1 x \, dx = \frac{1}{2} \quad (\text{Recall: } \int x \, dx = \frac{x^2}{2} + C) \]

we will get this in Sect. 5.3

Area \approx \frac{1}{2} \text{ base x height} = \frac{1}{2}
If $y = x^2$, then $y' = 2x$.

The derivative is the slope of the tangent at $x = a$.

Sections 5.1, 5.2, 5.4:

The definite integral $\int_a^b f(x) \, dx = \text{Area}$ between $y = f(x)$, $x$-axis, $x = a$, and $x = b$.
Section 5.3. Evaluating Definite Integrals.

"Definition." A DEFINITE INTEGRAL is a INDEFINITE INTEGRAL (Sec. 4.9) that you plug numbers into to get a number.

\[ \int f(x) \, dx = F(x) + C \]

\[ \int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a) \]

Do not bother forgetting on a C because it will be subtracted out anyway.

\[ a, b = \text{limits of integration} \]

\[ \text{lower limit} \quad \text{upper limit} \]

\[ f(x) = \text{integrand} \quad \text{("the inside")} \]

\[ F(x) \text{ (really } F(x) + C \text{) = antiderivative of } f(x) \]

\[ = F'(x) = f(x) \]
\[
\text{E.g., } \int_1^2 x^2 \, dx = \left. \frac{x^3}{3} \right|_1^2
\]

\[
= \frac{(2)^3}{3} - \frac{(1)^3}{3}
\]

\[
= \frac{8}{3} - \frac{1}{3}
\]

\[
= \frac{7}{3} \approx 2.333
\]

\[f(x) = x^2\]

= area between \( f(x) = x^2 \), \( x = 0 \) and \( x = 2 \)
Check:

**TI-83**

[MATH] → [fn Int] → [ENTER]

scroll down to

```
fnInt(

\[ \int_{1}^{2} x^{2} \, dx \]
```

Variable of integration

```
\int_{1}^{2} x^{2} \, dx
```

→ [ENTER]

2.3333333333

✓
TI-86

2nd \rightarrow \text{CALC} \rightarrow \text{Func Int } \rightarrow \\ f_{\text{Int}}(x) \\
\downarrow \\
f_{\text{Int}}(x^2, x, 1, 2) \\
\rightarrow \text{ENTER} \\
2, 333 333 333 33 33 \checkmark
\[ \int_{\pi}^{2\pi} \sin x \, dx = -\cos x \bigg|_{\pi}^{2\pi} = (-\cos 2\pi) - (-\cos \pi) = 1 - (-1) = 2 \]

The area is \(2\) square units.

The area is \(-2\) square units.

\[ \text{Area is } \begin{cases} \text{NEGATIVE when below } x\text{-axis} \\ \text{POSITIVE when above } x\text{-axis} \end{cases} \]
We will now go on to further examples. For each example,

SEE 1 Table of Antidifferentiation Formulas at bottom of p. 330, Sect. 4.9, text

2 Table of Indefinite Integrals on p. 372, Sect. 5.3, text
Examples.

1. Evaluate \( \int_{-2}^{4} (3x - 5) \, dx \).

\[
\int_{-2}^{4} (3x - 5) \, dx = 3 \cdot \int_{-2}^{4} x \, dx - 5 \cdot \int_{-2}^{4} 1 \, dx
\]

\[
= 3 \left( \frac{x^2}{2} \right) \bigg|_{-2}^{4} - 5 (x) \bigg|_{-2}^{4}
\]

\[
= \frac{3 \cdot 4^2}{2} - 5 \cdot 4 - \left( \frac{3 \cdot (-2)^2}{2} - 5 \cdot (-2) \right)
\]

\[
= (24 - 20) - (6 + 10)
\]

\[
= 4 - 16
\]

\[
= -12
\]
\[ \int_{-2}^{4} (3x - 5) \, dx = \text{net area between}\]
\[ \text{curve, x-axis, and lines } x = -2 \text{ and } x = 4\]
\[ = A_1 + (-A_2)\]
\[ = A_1 - A_2\]
\[ = -12\]
\[ Q. \quad \text{Evaluate } \int_{\pi/4}^{\pi/2} \sin t \, dt. \]

\[
\int_{\pi/4}^{\pi/2} \sin t \, dt = -\cos t \bigg|_{\pi/4}^{\pi/3} \\
= \left( -\cos \frac{\pi}{3} \right) - \left( -\cos \frac{\pi}{4} \right) \\
= -\cos \frac{\pi}{3} + \cos \frac{\pi}{4}
\]

**Exact**
\[
= - \frac{1}{2} + \frac{\sqrt{2}}{2}
\]

\[
\approx 0.2071
\]

**Calculator**

![Graph showing a right triangle with angle \( \frac{\pi}{9} = 60^\circ \) and \( \frac{\pi}{4} = 45^\circ \).]
Evaluate $\int_4^8 \frac{1}{x} \, dx$.

$\int_4^8 \frac{1}{x} \, dx = \ln 1x \bigg|_4^8$

$= \ln 8 - \ln 4$

$\approx 0.6931$

$\ln \left( \frac{8}{4} \right)$

$= \ln 2$

$\int_4^8 \frac{1}{x} \, dx$ represents the (net) area between the curve, x-axis, and lines $x = 4$ and $x = 8$.

$\approx 0.6931$
\[ \int_1^4 \left( \frac{1}{x^2} + \frac{1}{x^3} + \sqrt{x} \right) \, dx = ? \]

\[ \int_1^4 \left( \frac{1}{x^2} + \frac{1}{x^3} + \sqrt{x} \right) \, dx = \int_1^4 (x^{-2} + x^{-3} + x^{1/2}) \, dx \]

\[ = \left. \frac{x^{-2+1}}{-2+1} + \frac{x^{-3+1}}{-3+1} + \frac{x^{1/2+1}}{1/2+1} \right|_1^4 \]

\[ = \left. \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + \frac{x^{3/2}}{3/2} \right|_1^4 \]

\[ = \left. -\frac{1}{x} - \frac{1}{2x^2} + \frac{2}{3} x^{3/2} \right|_1^4 \]

\[ = \left[ \left( -\frac{1}{4} - \frac{1}{2(4)^2} + \frac{2}{3} (4)^{3/2} \right) \right] - \left[ \left( -\frac{1}{1} - \frac{1}{2(1)^2} + \frac{2}{3} (1)^{3/2} \right) \right] \]

\[ = \left( -\frac{1}{4} - \frac{1}{2} + \frac{16}{3} \right) - \left( -1 - \frac{1}{2} + \frac{2}{3} \right) \]

\[ = -\frac{3}{4} + \frac{16}{3} - (-\frac{3}{2} + \frac{2}{3}) \]

\[ = -\frac{3}{4} + \frac{16}{3} + \frac{3}{2} - \frac{2}{3} \]

\[ = \left( \frac{16}{3} + \frac{3}{2} - \frac{2}{3} \right) - \frac{3}{4} \]

\[ = \left( \frac{16}{3} + \frac{9}{6} - \frac{4}{6} \right) - \frac{3}{4} \]

\[ = \frac{17}{6} - \frac{3}{4} \]

\[ = \frac{34}{12} - \frac{9}{12} \]

\[ = \frac{25}{12} \]
\[ f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{\sqrt{x}} \]
\[ \int_1^9 \frac{1}{2x} \, dx = ? \]

\[ \int_1^9 \frac{1}{2x} \, dx = \frac{1}{2} \int_1^9 \frac{1}{x} \, dx \]

\[ \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot 1 \]

\[ = \frac{1}{2} \int_1^9 \frac{1}{x} \, dx \]

\[ = \frac{1}{2} \ln |x| \bigg|_1^9 \]

\[ = \left[ \frac{1}{2} \ln 9 - \frac{1}{2} \ln 1 \right] \]

\[ = \frac{1}{2} \ln 9 \]

\[ = \frac{1}{2} \ln 3^2 \]
$$-307-$$

$$= \frac{1}{2} \cdot 2 \ln 3$$

$$= \ln 3$$

$$\approx 1.0986$$

$$y = \frac{1}{2x}$$

\[\text{AREA}\]
Section 5.3. Evaluating the Definite Integral.

Up to this point, we have shown the following:

Let \( t = \text{time} \)

\( s(t) = \text{distance as a function of time} \)

\( v(t) = \text{velocity as a function of time} \)

We know that

\( s'(t) = v(t) \).

Now, we found that

\[
\int_a^b v(t) \, dt = s(b) - s(a)
\]

where \( s'(t) = v(t) \)
(2) \( F(x) \) is called the antiderivative of \( f(x) \) when \( F'(x) = f(x) \) (i.e., the derivative of \( F \) is equal to \( f \)).

\[
\frac{d}{dx} x^3 = 3x^2 \\
\int 3x^2 \, dx = x^3 \\
3x^2 = \text{derivative of } x^3 \\
x^3 = \text{antiderivative of } 3x^2
\]

(3) \( \int_a^b f(x) \, dx \) is called the definite integral of \( f(x) \) and \( a \) and \( b \) are called the (lower and upper) limits of integration.

One can write

\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a)
\]

where \( F(x) \) is the antiderivative of \( f(x) \).
We will 

**GENERALIZE** this:

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

where \( F'(x) = f(x) \) over \([a, b]\)

**NOTES:**

1. We never actually proved that

\[
\text{AREA under } y = v(t) \text{ over } [a, b], \int_{a}^{b} v(t) \, dt = \left\{ \begin{array}{c}
\text{TOTAL DISTANCE} \\
\text{traveled from } t = a \\
\text{to } t = b, \quad s(b) - s(a)
\end{array} \right. 
\]

**SEE my HANDOUT on a rough justification that**

\[
\text{AREA under } y = f(x) \text{ over } [a, b], \int_{a}^{b} f(x) \, dx = \left\{ \begin{array}{c}
F(b) - F(a) \\
\text{where } F'(x) = f(x) \\
\text{on } [a, b]
\end{array} \right. 
\]

There is a proof in Sect. 5.3 of the text. They make reference to the "Mean Value Theorem," which was discussed in Sect. 4.3. We never covered this theorem, so do not worry about it.
\[ E \text{, 9.} \int_{-1}^{1} 3x^2 \, dx = x^3 \Bigg|_{-1}^{1} = (1)^3 - (-1)^3 \]

\[ \frac{dx}{dx}(x^3) = 3x^2 \]

\[ = 1 + (-1) = 1 - 1 = \frac{2}{2} \]

**Given** \( f(x) = 3x^2 \), **make an educated guess to find** \( F(x) \). So, ask, \( "What function \ F(x) \ do \ I \ need \ such \ that \ its \ derivative \ equals \ f(x) = 3x^2. \" \)

\( 3x^2 \) looks like it comes from taking the **derivative** of \( x^3 \).

Does it? **Try it**;

\( F(x) = x^3 \Rightarrow F'(x) = \frac{d}{dx}(x^3) = 3x^2 = f(x) \checkmark \)

So \( F(x) = x^3 \).

\( (t) \int f(x) \, dx \), **without the limits of integration**, is called the **indefinite integral** of \( f(x) \). It will **just equal** \( F(x) \) (plus a constant, as we will see) and **not be evaluated at** \( a \) and \( b \). So

\[ \int f(x) \, dx = F(x) + C \]
where $F'(x) = f(x)$ and $C$ is called a constant of integration. The reason why we add on a $C$ is because the derivative of $F(x) + C$ is still $f(x)$:

$$\frac{d}{dx} [F(x) + C] = \frac{d}{dx} F(x) + \frac{d}{dx} C = F'(x) + 0$$

Defined as $\int f(x) \, dx = F(x) + C$

Integrates $F(x) + 1$
$F(x) + \pi$
$F(x) + 10$
$F(x) - \frac{1}{2}$
etc.
etc.

$F(x) + C$ is the most general function you can have whose derivative equals $f(x)$. 

---
Formulas to Make Your "Educated Guesses" About F(x) Easier

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \quad \text{...} \]

(Otherwise, we would have \( \frac{x^{0+1}}{0+1} = \frac{x^0}{0} = \frac{1}{0} \))

\[ \int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C, \quad x \neq 0 \quad \text{...} \]

\[ \frac{d}{dx} \ln |x| = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases} \quad \text{CHAIN RULE} \]

\[ \frac{d}{dx} \ln (g(x)) = \frac{1}{g(x)} \cdot g'(x) \]

\[ \int e^x \, dx = e^x + C \]

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]
\[
\int \sec^2 x \, dx = \tan x + C \\
\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C
\]

MEMORIZE ALL OF THESE FOR MATH 201 !!!

(SEE Table on p. 534, Sect. 4.9, or Table on p. 375, Sect. 5.3)
Examples.

1. (HW Exercise 7, p. 380.)

Evaluate \( \int_{-2}^{4} (3x - 5) \, dx \).

\[
\int_{-2}^{4} (3x - 5) \, dx = \int_{-2}^{4} 3x \, dx - \int_{-2}^{4} 5 \, dx
\]
\[
= \int_{-2}^{4} 3x \, dx - 5 \int_{-2}^{4} 1 \, dx
\]
\[
= 3 \left[ \frac{x^{1+1}}{1+1} \right]_{-2}^{4} - 5 \left[ \frac{x^{0+1}}{0+1} \right]_{-2}^{4}
\]
\[
= 3 \left[ \frac{x^2}{2} \right]_{-2}^{4} - 5 \left[ x \right]_{-2}^{4}
\]
\[
= 3 \left[ \frac{(4)^2 - (-2)^2}{2} \right] - 5 \left[ (4) - (-2) \right]
\]
\[
= 3 (16 - 4) - 5 (4 + 2)
\]
\[
= 18 - 30
\]
\[
= -12
\]

Note: Could have alternatively used the property
\[ \int_a^b c \, dx = c(b-a) \]

To integrate \( \int_{-2}^4 5 \, dx \):

\[ \int_{-2}^4 5 \, dx = 5 \left( 4 - (-2) \right) = 5(4+2) = 30 \]
2. (HW Exercise 9, p. 380.)

Evaluate \( \int_{-1}^{0} (2x - e^x) \, dx \).

\[
\int_{-1}^{0} (2x - e^x) \, dx = 2 \int_{-1}^{0} x \, dx - \int_{-1}^{0} e^x \, dx
\]

\[
= 2 \left[ \frac{x^2}{2} \right]_{-1}^{0} - \left[ e^x \right]_{-1}^{0}
\]

\[
= 2 \left[ \frac{0^2}{2} - \frac{(-1)^2}{2} \right] - (e^0 - e^{-1})
\]

\[
= 2 \left( -\frac{1}{2} \right) - (1 - e^{-1})
\]

\[
= -1 - 1 + \frac{1}{e}
\]

\[
= \frac{1}{e} - 2
\]
3. (HW Exercise 15, p. 380.)

Evaluate \( \int_0^1 u \left( \sqrt{u} + 2\sqrt[3]{u} \right) \, du \).

\[
\int_0^1 u \left( \sqrt{u} + 2\sqrt[3]{u} \right) \, du
\]

\[= \int_0^1 u^{1/2} + u^{2/3} \, du\]

\[= \int_0^1 \left( u^{1 + \frac{1}{2}} + u^{1 + \frac{1}{3}} \right) \, du\]

\[= \int_0^1 \left( u^{3/2} + u^{4/3} \right) \, du\]

\[= \int_0^1 u^{3/2} \, du + \int_0^1 u^{4/3} \, du\]

\[= \left[ \frac{u^{3/2} + 1}{3/2 + 1} \right]_0^1 + \left[ \frac{u^{4/3} + 1}{4/3 + 1} \right]_0^1\]

\[= \frac{1}{3/2 + 1} + \frac{1}{4/3 + 1}\]

\[= \frac{1}{3/2 + 1} + \frac{1}{4/3 + 1}\]

\[= \frac{2}{3} + \frac{3}{2} = \frac{5}{2}\]

\[= \frac{5}{2}\]
\[
\sqrt{\frac{u^{5/2}}{\frac{5}{2}}} + \left[ \frac{u^{4/3}}{\frac{7}{3}} \right]_0^n
= \left[ \frac{2}{5} u^{5/2} \right]_0^n + \left[ \frac{3}{7} u^{4/3} \right]_0^n
= \left[ \frac{2}{5} (1)^{5/2} - \frac{2}{5} (0)^{5/2} \right]
+ \left[ \frac{3}{7} (1)^{4/3} - \frac{3}{7} (0)^{4/3} \right]
= \frac{2}{5} + \frac{3}{7}
= \frac{2 \cdot 7}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 5}
= \frac{14}{35} + \frac{15}{35}
= \frac{29}{35}
\]
Evaluate \( \int_{\pi/4}^{\pi/3} \sin t \, dt \).

\[
\int_{\pi/4}^{\pi/3} \sin t \, dt = \left[-\cos t\right]_{\pi/4}^{\pi/3}
\]

\[
= (-\cos \frac{\pi}{3}) - (-\cos \frac{\pi}{4})
\]

\[
= \frac{\cos \frac{\pi}{4} - \cos \frac{\pi}{3}}{1}
\]

**Calculator**: \( \approx 0.2071 \)

**Exact**: \( \frac{\sqrt{2}}{2} - \frac{1}{2} \)
Evaluate \( \int_{-e^2}^{-e} \frac{3}{x} \, dx \).

\[
\int_{-e^2}^{-e} \frac{3}{x} \, dx = 3 \int_{-e^2}^{-e} \frac{1}{x} \, dx
= 3 \left[ \ln |x| \right]_{-e^2}^{-e}
= 3 \left( \ln |e| - \ln |e^2| \right)
= 3 \left( \frac{\ln e}{1} - \frac{\ln e^2}{2} \right)
= 3 (1 - 2)
= -3
\]

Will obtain same result if use \( 3 \times (\ln e^1 - \ln e^2) \).
6. (HW Exercise 25, p. 380)

Evaluate \[ \int_{1}^{\sqrt{3}} \frac{6}{1 + x^2} \, dx. \]

\[
\int_{1}^{\sqrt{3}} \frac{6}{1 + x^2} \, dx = 6 \int_{1}^{\sqrt{3}} \frac{1}{1 + x^2} \, dx
\]

\[
= 6 \cdot \left[ \tan^{-1} x \right]_{1}^{\sqrt{3}}
\]

\[
= 6 \left[ \tan^{-1}(\sqrt{3}) - \tan^{-1}(1) \right]
\]

CALCULATOR:

\approx 1.5708

EXACT:

\[
= 6 \left( \frac{\pi}{3} - \frac{\pi}{4} \right)
\]

since \( \tan \frac{\pi}{3} = \frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} \)

\[
= \frac{\sqrt{3}}{2} \cdot \frac{2}{1}
\]

\[
= \frac{\sqrt{3}}{1}
\]

and \( \tan \frac{\pi}{4} = \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} \)
\[
\begin{align*}
\text{Area} &= \frac{\pi}{2} \\
&= \frac{\sqrt{2}}{2} \\
&= 1 \\
&= 6 \left( \frac{\pi}{4} - \frac{\pi}{4} \right) \\
&= 6 \left( \frac{\pi}{12} - \frac{\pi}{12} \right) \\
&= 6 \left( \frac{\pi}{12} \right) \\
&= \frac{\pi}{2}
\end{align*}
\]
Lecture

Section 5.4. The Fundamental Theorem of Calculus [I and II]

These are actually two parts to this theorem. We actually had exposure to one of them in Section 5.4.

First, a couple of important properties we need to know.
Some More Properties of $\int_a^b f(x) \, dx$

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]

\[ \Delta x = \frac{b-a}{n} \quad -\Delta x = \frac{a-b}{n} \]

\[ \frac{a}{\Delta x} \quad \frac{b}{-\Delta x} \]

Example: \[ \int_0^5 x^2 \, dx = -\int_5^0 x^2 \, dx \]

\[ \left[ \frac{x^3}{3} \right]_0^5 - \left[ \frac{x^3}{3} \right]_5^0 \]

\[ \frac{125}{3} - \frac{27}{3} \]

\[ \frac{125}{3} - \frac{27}{3} \]

\[ \frac{125}{3} - \frac{27}{3} \]
b. If \( m \leq f(x) \leq M \) on \([a, b]\)

Then

\[
m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)
\]

(comes from Property 2, p. 384, text, and from property that \( \int_a^b c \, dx = c(b-a) \)).

E.g., (HW Exercise 4, p. 390) Use Property 3 to estimate the value of the integral

\[
\int_0^2 \sqrt{x^2+1} \, dx
\]

**CALCULATOR graph to find**

- \( m = \) lowest point of \( \sqrt{x^2+1} \) over \([0, 2]\)
- \( M = \) highest point of \( \sqrt{x^2+1} \) over \([0, 2]\)
m and M are at the endpoints of [0, 2]. So

\[
m = f(0) = \sqrt{(0)^2 + 1} = \sqrt{1} = 1
\]
\[
M = f(2) = \sqrt{(2)^2 + 1} = \sqrt{9} = 3
\]

\[
m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \quad \Rightarrow
\]
\[
1 \cdot (2-0) \leq \int_0^2 \sqrt{x^2 + 1} \, dx \leq 3 \cdot (2-0) \quad \Rightarrow
\]
\[
2 \leq \int_0^2 \sqrt{x^2 + 1} \, dx \leq 6
\]
THE FUNDAMENTAL THEOREM OF CALCULUS, PART I (FTC I)

As long as \( f(x) \) is continuous on \([a, b]\)
then you can take the integral of \( f(x) \)
from \( a \) to \( b \),

\[
\int_a^b f(x) \, dx
\]

change \( f(x) \, dx \) to \( f(t) \, dt \) and then replace
the upper limit \( b \) by the variable \( x \) between \( a \) and \( b \),

\[
\int_a^x f(t) \, dt
\]

and you will end up with a NEW
FUNCTION, say, \( g(x) \):

\[
g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b
\]

A NEW FUNCTION

E.g., Consider \( f(x) = \frac{1}{x} \) on \([1, 10]\).
Is \( f \) cont. on \([0, 17]\)?
YES, \( f(x) = \frac{1}{x} \) is cont. on \([1, 10]\)

\[
\int_{1}^{10} \frac{1}{x} \, dx \rightarrow \int_{1}^{X} \frac{1}{t} \, dt = \left[ \ln |t| \right]_{1}^{X}
\]

\[
= \ln X - \ln 1
\]

\[
= \ln X - 0
\]

\[
= \ln X
\]

Our new function \( g(x) \) is:

\[
g(x) = \ln x, \quad 1 \leq x \leq 10
\]

NOTICE: \( g'(x) = \frac{1}{x} = f(x) \). This leads us to the next thing.
Not only do we have a new function
\[ g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b, \]
but \( g(x) \) is DIFFERENTIABLE and
\[ g'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x), \quad a \leq x \leq b. \]

This says two things:

1. \( g'(x) = f(x) \implies g \) is the ANTIDERIVATIVE of \( f \) on \([a, b]\).

2. \( \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \implies \) differentiation "UNDOES" integration and you get back \( f \) (but \( f(x) \) now, not \( f(t) \)). Here the "\( a \)" can be anything and does not matter.

Examples. Use FTC I to find the derivative of the following functions.
1. (HW Exercise 11, p. 390.)

$$g(x) = \int_{1}^{x} (t^2 - 1)^{20} \, dt$$

Two ways to do:

1. $$g(x) = \int_{1}^{x} (t^2 - 1)^{20} \, dt$$  \[
\Rightarrow \quad \frac{dg}{dx} = f(x) = \boxed{(x^2 - 1)^{20}}
\]

2. $$g'(x) = \frac{d}{dx} \int_{1}^{x} (t^2 - 1)^{20} \, dt$$  \[
\Rightarrow \quad (t^2 - 1)^{20} \quad \text{cancel the} \, \frac{dx}{dt} \text{and the} \, \int_{1}^{x}
\]

change the

$$t \text{ to } \boxed{\text{on } x}$$
(HW Exercise 14, p. 390.)

\[ F(x) = \int_x^2 \cos(t^2) \, dt \]

**BEFORE ANYTHING, you must make sure the \( x \) is an **UPPER**, NOT LOWER, LIMIT:**

\[ F(x) = \int_x^2 \cos(t^2) \, dt \quad \leftarrow - \int_2^x \cos(t^2) \, dt \]

\[ \text{switch limits and} \]

\[ \text{add a negative} \]

\[ = \int_2^x -\cos(t^2) \, dt \]

\[ \text{pull the negative inside} \]

\[ \text{the integral to be part of the } -\cos(t^2) \]

Then

\[ F(x) = \int_2^x \cos(t^2) \, dt \quad \Rightarrow \]

\[ f(t) \]

\[ F'(x) = f(x) = \boxed{\cos(x^2)} \]
\( F'(x) = \frac{d}{dx} \int_2^x -\cos(t^2) \, dt \rightarrow -\cos(x^2) \)
As long as $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

or

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

or

$$\left[ \frac{d}{dx} F(x) \right]_a^b = F(b) - F(a)$$

This time, integration "UNDOES" differentiation and you get back $F(x)$, evaluated at $a$ and $b$.
FTC I: \( \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \)

FTC II: \( \int_{a}^{x} \frac{d}{dx} F(x) \, dx = [F(x)]^{b}_{a} \)
Section 5.5 The Substitution Rule.

This rule handles some difficult to integrate functions (that can usually be viewed as the composition of two functions multiplied by another function). It is the CHAIN RULE in reverse.

\[
\int g'(x) \, f(g(x)) \, dx = F(g(x)) + C
\]

where \( F'(x) = f(x) \) (and so \( \frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) \)).

To handle integrals like

\[
\int g'(x) \, f(g(x)) \, dx
\]

we will not use the FORMULA but instead do the following:

Let \( u = g(x) \). Then \( \frac{du}{dx} = g'(x) \).
Substitute \( u \) for \( g(x) \) and \( \frac{du}{dx} \) for \( g'(x) \) to obtain a new, easier to integrate, integral

\[
\int \frac{du}{dx} f(u) \, dx = \int f(u) \, \frac{du}{dx} \, dx
\]

Treat \( \frac{du}{dx} \) like a fraction and cancel the \( dx \)'s

\[
= \int f(u) \, du
\]

= \( F(u) + C \)

Substitute back the \( g(x) \) for the \( u \)

= \( F(g(x)) + C \)
Examples.

1. \[ \int 2x e^{x^2} \, dx = ? \]

Let \( u = x^2 \). Then \( \frac{du}{dx} = 2x \). So

\[ \int 2x e^{x^2} \, dx = \int \frac{du}{dx} e^u \, dx \]

\[ = \int e^u \, du \]

\[ = e^u + C \]

\[ = e^{x^2} + C. \]
2. \[ \int 3x^2(x^3+1)^{10} \, dx = ? \]

Let \( u = x^3 + 1 \). Then \( \frac{du}{dx} = 3x^2 \). So, \[ \int 3x^2(x^3+1)^{10} \, dx = \int \frac{du}{dx} u^{10} \, dx \]

\[ = \int u^{10} \frac{du}{dx} \, dx \]

\[ = \int u^{10} \, du \]

\[ = \frac{u^{11}}{11} + C \]

\[ = \frac{(x^3+1)^{11}}{11} + C \]