CMSC691 Optimization, VCU
Fall 2016, Assignment 2: Convex Functions

Due: Friday, October 14, at 11:59 PM.
Submit via Blackboard under Assignments tab.

Total marks: 50 marks + 5 marks bonus. For full marks, you must show your work.

1 Exercises

1. (10 marks) The following questions ask for a characterization of functions f satisfying the desired criteria.
   (a) (5 marks) When is the epigraph of a function a halfspace?
   (b) (5 marks) When is the epigraph of a function a convex cone? (Hint: The hard part is understanding when the epigraph is a cone. For this, consider the set of functions which satisfy $f(\theta x) = \theta f(x)$ for any $\theta \geq 0$.)
   (c) (Bonus: 5 marks) When is the epigraph of a function a polyhedron?

2. (5 marks) Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex with $\operatorname{dom} f = \mathbb{R}^n$, and bounded above on $\mathbb{R}^n$ (i.e. there exists constant c such that $f(x) \leq c$ for all $x \in \operatorname{dom} f$). Show that f is constant. You may assume f is differentiable.

3. (5 marks) In class, we stated *second-order* conditions for a function to be convex, but did not prove them. In this question, you will prove these conditions for the 1-dimensional case (i.e. you will finally prove what you have assumed all these years in your high school education). Specifically, show that a twice differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ is convex if and only if its domain is convex and $f''(x) \geq 0$ for all $x \in \operatorname{dom} f$.

   Hints:
   - For both directions of the proof, use the first-order condition for convexity derived in class as a given.
   - For one direction of the proof, start by expressing f to first order plus a second-order remainder term using Taylor's theorem (look it up online!). It will be helpful to use the *Lagrange* form of the remainder term.)
   - For the other direction of the proof, use the first-principles definition $f''(x) = \lim_{h \to 0}(f'(x+h) - f'(x))/h$.

4. (15 marks) For each of the following functions, determine whether it is convex, concave, or neither (i.e. give a proof). Use the fact that if the Hessian of a function is positive semidefinite, negative semidefinite, or indefinite, then f is convex, concave, and neither, respectively. (A symmetric matrix is *negative semidefinite* if it has only non-positive eigenvalues. It is *indefinite* if it has both positive and negative eigenvalues.)
   (a) (5 marks) $e^x - 1$ on $\mathbb{R}$.
   (b) (5 marks) $x_1 x_2$ on $\mathbb{R}^2_{++}$.
   (c) (5 marks) $1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$. (Hint: Compute the Hessian, and then use the Schur complement!)

5. (5 marks) Suppose $p < 1$, $p \neq 0$. Show that

$$f(x) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

   with $\operatorname{dom} f = \mathbb{R}^n_{++}$ is concave. This includes as a special case the *harmonic mean* $f(x) = (\sum_{i=1}^{n} 1/x_i)^{-1}$. (Hint: Adapt the proofs for the log-sum-exp function and the geometric mean in Section 3.1.5 of the text. The former of these we did together in class.)

6. (10 marks) In general, the product or ratio of two convex functions is *not* convex. However, there are some results that apply to functions in $\mathbb{R}$. Prove the following.

   (a) (5 marks) If f and g are convex, both non-decreasing (or non-increasing), and non-negative functions, then fg is convex.

   (b) (5 marks) If f is convex, non-decreasing, and positive, and g is concave, non-increasing, and positive, then f/g is convex.