CMSC491 Introduction to Quantum Computation, VCU
Fall 2015, Assignment 7

Due: Tuesday, November 3 at start of class

Total marks: 20 marks + 10% bonus for typing your solutions in LaTeX.

1 Exercises

1. (6 marks) Let U be a unitary operator.
   (a) (4 marks) Prove that all the eigenvalues of U have the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$. (Hint: Use the spectral decomposition, and recall the polar form for complex numbers.)
   (b) (2 marks) Why can we rewrite the statement above by assuming that all eigenvalues have the form $e^{2\pi i\theta}$ for $\theta \in [0, 1)$?
2. (8 marks) Let U be a unitary operator whose eigenvalues all have form $e^{2\pi i\theta}$ for θ requiring only m bits of precision, i.e. $\theta = j/2^m$. Assume furthermore that all eigenvalues of U are distinct.
   (a) (2 marks) Let $|\psi\rangle$ be an eigenvector of U with eigenvalue $e^{2\pi i\theta}$. Given input state $|0\rangle^{\otimes m}|\psi\rangle$, what quantum state does the phase estimation algorithm from class output? (Do not discard any registers in your answer.)
   (b) (4 marks) Let $\{|\psi_i\rangle\}$ be a complete set of eigenvectors for U with corresponding eigenvalues $\left\{e^{2\pi i\theta_i}\right\}$. Let
   $$|\phi\rangle = \sum_i \alpha_i |\psi_i\rangle$$
   be a unit vector. Given input state $|0\rangle^{\otimes m}|\phi\rangle$, what quantum state does the phase estimation algorithm from class output? (Hint: No need to go into the detailed analysis for the algorithm; rather, think at a higher level and recall that the algorithm is one big unitary transformation.)
   (c) (2 marks) Continuing with the question from part (b), if at the end we measure the first register in the standard basis, what distribution over eigenvalues of U do we obtain? In other words, with what probability will we find phase θ_i in the first register?
3. (6 marks)
   (a) (3 marks) Draw the circuit for $\widetilde{\mathrm{QFT}}_4$ from class.
   (b) (3 marks) Run this circuit on input $|01\rangle$ to show that it maps $|01\rangle$ to
   $$\frac{1}{2}\sum_{k=0}^{3} e^{\pi i k/2}|k_0 k_1\rangle,$$
   where as done in class, k_0k_1 is k with its bits reversed, i.e. $k = k_1k_0$.