CMSC491 Introduction to Quantum Computation, VCU
Fall 2015, Assignment 5

Due: Tuesday, October 6 at start of class

Total marks: 20 marks + 10% bonus for typing your solutions in LaTeX.

1 Exercises

1. (4 marks) This question will practice working with observables.

   (a) (2 marks) Define $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, and consider projective measurement $M = \{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$ with labels corresponding to outcomes $S = \{1, -1\}$, respectively. Write down the matrix $C \in \mathcal{L}(\mathbb{C}^2)$ corresponding to the observable for this measurement.

   (b) (2 marks) Suppose state $|0\rangle \in \mathbb{C}^2$ is measured via C. What is the expected value for the measurement?

2. (8 marks) In this question, we consider how well the CHSH game strategy from class fares if we use a *less* entangled state as a shared resource between Alice and Bob. Specifically, imagine we use the same observables as before, but now we replace $|\Phi^+\rangle$ as a shared state with $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. Intuitively, as α gets closer to 1, this state becomes less entangled, and for $\alpha = 1$, it becomes a product state (i.e. non-entangled).

   (a) (6 marks) What is the probability of winning the CHSH game with shared state $|\psi\rangle$? (Hint: Similar to the Lecture 5 notes, begin by arguing that for *any* $|\psi\rangle$, the quantity $\mathrm{Tr}(A \otimes B|\psi\rangle\langle\psi|) = \Pr(\text{output same bits}) - \Pr(\text{output different bits})$, i.e. the interpretation of this quantity does not depend on our choice of $|\psi\rangle$.)

   (b) (2 marks) Based on your answer above, what is the probability of Alice and Bob winning with this strategy if $\alpha = 1$, i.e. $|\psi\rangle$ is unentangled?

3. (8 marks) As seen in class, Deutsch's algorithm is able to determine with certainty (i.e. probability 1) whether a 1-bit function is constant or balanced. Suppose you wish to run Deutsch's algorithm in your lab, but your equipment doesn't quite function as you expect. In particular, instead of preparing your desired initial state of $|0\rangle|1\rangle$ to the algorithm, your machine prepares state $|\psi\rangle|1\rangle$ for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

   (a) (6 marks) Assuming the function f is balanced, what is the probability that Deutsch's algorithm on your faulty initial state $|\psi\rangle|1\rangle$ will correctly output "balanced", i.e. that the final measurement result in the algorithm has label 1?

   (b) (2 marks) There is only so much "error" that the algorithm can tolerate before it becomes useless — for which range of the parameter $0 \leq |\alpha| \leq 1$ is the probability of success in part 3a less than or equal to 1/2? In other words, for which values of $|\alpha|$ is it better to forget about running Deutsch's algorithm and instead just flip a classical (unbiased) coin to "decide" if f is balanced?

4. (bonus, 4 marks) Show that a bipartite pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is entangled if and only if $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ has rank 2. In other words, show that the Schmidt rank of $|\psi\rangle$ equals the rank of ρ_A. (Hint: Start by writing out the Schmidt decomposition of $|\psi\rangle$. When taking the partial trace, use the fact that $\mathrm{Tr}_B(C_A \otimes D_B) = C_A \cdot \mathrm{Tr}(D_B)$.)