CMSC 303 Introduction to Theory of Computation, VCU
Fall 2017, Assignment 6
Due: Tuesday, November 28, 2017 in class

Total marks: 40 marks + 4 marks bonus for typing your solutions in LaTeX.

Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0, 1\}$. This assignment will get you primarily to practice reductions in the context of decidability.

1. [10 marks] We begin with some mathematics regarding uncountability. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers.
   (a) [5 marks] Prove that the set of all perfect squares $S = \{0, 1, 4, 9, 16 \ldots\}$ has the same size as $\mathbb{N}$ by giving a bijection between S and $\mathbb{N}$.
   (b) [5 marks] Let B denote the set of all infinite sequences over $\{0, 1\}$. Show that B is uncountable using a proof by diagonalization.
2. [9 marks] We next move to a warmup question regarding reductions.
   (a) [2 marks] Intuitively, what does the notation $A \leq B$ mean for problems A and B?
   (b) [2 marks] What is a mapping reduction $A \leq_m B$ from language A to language B? Give both a formal definition, and a brief intuitive explanation in your own words.
   (c) [2 marks] What is a computable function? Give both a formal definition, and a brief intuitive explanation in your own words.
   (d) [3 marks] Suppose $A \leq_m B$ for languages A and B. Please answer each of the following with a brief explanation.
      i. If B is decidable, is A decidable?
      ii. If A is undecidable, is B undecidable?
      iii. If B is undecidable, is A undecidable?
3. [21 marks] Prove using reductions that the following languages are undecidable. (Do not use Rice's theorem as a black box!)
   (a) [5 marks] Show via a reduction from the halting problem that

   $$L = \{\langle M, w\rangle \mid M \text{ is a TM which does not halt on input } w\}$$

   is undecidable.
   (b) [8 marks] $L = \{\langle M\rangle \mid M \text{ is a TM and } \epsilon \in L(M)\}$.
   (c) [8 marks] Consider the problem of determining whether a TM M on an input w ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.