Conditional Posterior Cramér–Rao Lower Bounds for Nonlinear Sequential Bayesian Estimation

Long Zuo, Ruixin Niu, *Member, IEEE*, and Pramod K. Varshney, *Fellow, IEEE*

***Abstract*—The posterior CramérRao lower bound (PCRLB) for sequential Bayesian estimators, which was derived by Tichavsky *et al.* in 1998, provides a performance bound for a general nonlinear filtering problem. However, it is an offline bound whose corresponding Fisher information matrix (FIM) is obtained by taking the expectation with respect to all the random variables, namely the measurements and the system states. As a result, this unconditional PCRLB is not well suited for adaptive resource management for dynamic systems. The new concept of conditional PCRLB is proposed and derived in this paper, which is dependent on the actual observation data up to the current time, and is implicitly dependent on the underlying system state. Therefore, it is adaptive to the particular realization of the underlying system state and provides a more accurate and effective online indication of the estimation performance than the unconditional PCRLB. Both the exact conditional PCRLB and its recursive evaluation approach including an approximation are derived. Further, a general sequential Monte Carlo solution is proposed to compute the conditional PCRLB recursively for nonlinear non-Gaussian sequential Bayesian estimation problems. The differences between this new bound and existing measurement dependent PCRLBs are investigated and discussed. Illustrative examples are also provided to show the performance of the proposed conditional PCRLB.**

***Index Terms*—Kalman filters, nonlinear filtering, particle filters, posterior Cramér–Rao lower bounds.**

I. Introduction

THE conventional Cramér–Rao lower bound (CRLB) [2] on the variance of estimation error provides the performance limit for any unbiased estimator of a fixed parameter. For a random parameter, Van Trees presented an analogous bound, the posterior CRLB (PCRLB) [2], which is also referred to as the Bayesian CRLB. The PCRLB is defined as the inverse of the Fisher information matrix (FIM) for a random vector and provides a lower bound on the mean-square error (MSE) of any estimator of the random parameter, which in general is a vector. In [1], Tichavsky *et al.* derived an elegant recursive approach to calculate the sequential PCRLB for a general multi-dimensional discrete-time nonlinear filtering problem.

The PCRLB is a very important tool, since it provides a theoretical performance limit of any estimator for a nonlinear filtering problem under the Bayesian framework. In an unconditional PCRLB, the FIM is derived by taking the expectation with respect to the joint distribution of the measurements and the system states up to the current time. As a result, the very useful measurement information is averaged out and the unconditional PCRLB becomes an offline bound. It is determined only by the system dynamic model, system measurement model and the prior knowledge regarding the system state at the initial time, and is thus independent of any specific realization of the system state, as we will show later in the paper. As a result, the unconditional PCRLB does not reflect the nonlinear filtering performance for a particular system state realization very faithfully. This is especially true when the uncertainty in the state model (or equivalently the state process noise) is high and thus the prior knowledge regarding the system state at the initial time quickly becomes irrelevant as the system state evolves over time.

Some attempts have been made in the literature to include the information obtained from measurements by incorporating the tracker's information into the calculation of the PCRLB. In [3], a renewal strategy has been used to restart the recursive unconditional PCRLB evaluation process, where the initial time is reset to a more recent past time, so that the prior knowledge of the initial system state is more useful and relevant to the sensor management problem. The resulting PCRLB is, therefore, conditioned on the measurements up to the reset initial time. Based on the PCRLB evaluated in this manner, a sensor deployment approach is developed to achieve better tracking accuracy which at the same time uses the limited sensor resources more efficiently. This approach is extended in [4] to incorporate sensor deployment and motion uncertainties, and to manage sensor arrays for multi-target tracking problems in [5] and [6]. In the renewal strategy proposed in [3], using a particle filter, the posterior probability density function (PDF) of the system state at the reset initial time is represented nonparametrically by a set of random samples (particles), from which it is difficult to derive the exact Fisher information matrix. One solution is to use a Gaussian approximation, and in this case the FIM at the reset initial time can be taken as the inverse of the empirical covariance matrix estimated based on the particles. This, however, may incur large errors and discrepancy, especially in a highly nonlinear and non-Gaussian system. Once restarted, the renewal based approach recursively evaluates the PCRLB as provided in [1] until the next restart. Since the FIM at the reset initial

Manuscript received February 03, 2010; accepted September 10, 2010. Date of publication September 23, 2010; date of current version December 17, 2010. This work was presented in part at the International Conference on Information Fusion, Seattle, Washington, July 2009. This work was supported in part by U.S. Army Research Office under Grant W911NF-09-1-0244, and by U.S. Air Force Office of Scientific Research under Grant FA9550-10-1-0263. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Ta-Hsin Li.

The authors are with Syracuse University, Department of Electrical Engineering and Computer Science, Syracuse, NY 13244 USA (e-mail: lzuo@syr.edu; rniu@syr.edu; varshney@syr.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2010.2080268

time is evaluated based on filtering results rather than the previous FIM, this is not an entirely recursive approach. In contrast, in this paper, we introduce the notion of conditional PCRLB, which is shown to be different from the PCRLB based on renewal strategy presented in [3], through analysis and numerical examples. A systematic recursive approach to evaluate the conditional PCRLB with approximation is also presented.

Another related work is reported in [7], where a PCRLB based adaptive radar waveform design method for target tracking has been presented. In [7], for a system with a linear and Gaussian state dynamic model, but nonlinear measurement model, the framework of the unconditional recursive PCRLB derived in [1] has been retained. Only one term corresponding to the contribution of the future measurement to the FIM has been modified in an ad hoc manner to include the measurement history, by taking the expectation of the second-order derivative of the log-likelihood function with respect to the joint probability density function (PDF) of the state and measurement at the next time step conditioned on the measurements up to the current time. The heuristically modified PCRLB calculated in this manner does not yield the exact conditional PCRLB, as shown later in this paper.

In [8], for nonlinear target tracking problems, an algorithm is developed to select and configure radar waveforms to minimize the predicted MSE in the target state estimate, which is the expectation of the squared error over predicted states and observations given a past history of measurements. The predicted MSE is computationally intractable, and in [8] it has been approximated by the covariance update of the unscented Kalman filter.

Given the importance of the PCRLB based adaptive sensor management problem, to take advantage of the available measurement information, we have systematically developed the exact conditional PCRLB based on first principles. The proposed conditional PCRLB is dependent on the past data and hence implicitly dependent on the system state. The conditional PCRLB provides a bound on the conditional MSE of the system state estimate, based on the measurements up to the current time. In this paper, we systematically derive an approximate recursive formula to calculate the conditional PCRLB for nonlinear/non-Gaussian Bayesian estimation problems. The cumulative error due to the approximation is not severe even for a highly nonlinear problem, as demonstrated in a simulation example. Further, we present numerical approximation approaches for the computation of the recursive formula through particle filters. Since the conditional PCRLB is a function of the past history of measurements, which contains the information of the current realization of the system state, an approach based on it is expected to lead to much better solutions to the sensor resource management problem than those based on the unconditional PCRLB.

The paper is organized as follows. In Section II, background on the conventional PCRLB and its recursive evaluation are presented. The conditional PCRLB dependent on the past measurement history is derived and a recursive approach to evaluate it is provided in Section III. The differences between the proposed conditional PCRLB and the existing measurement dependent PCRLBs are investigated and discussed. A general sequential Monte Carlo solution for computing the conditional PCRLB for nonlinear non-Gaussian sequential Bayesian estimation problems is developed in Section IV. Some simulation experiments are carried out to corroborate the theoretical results in Section V. Finally, concluding remarks are presented in Section VI.

II. Conventional PCRLB

We are interested in estimating the state $\mathbf{x}$ given the observation $\mathbf{z}$, where $\mathbf{x}$ and $\mathbf{z}$ are both random vectors with dimensions n_x and n_z respectively, $n_x, n_z \in \mathbb{N}$, and $\mathbb{N}$ is the set of natural numbers. Let $\hat{\mathbf{x}}(\mathbf{z})$ be an estimator of $\mathbf{x}$, which is a function of $\mathbf{z}$. The Bayesian Cramér-Rao inequality [2] shows that the mean squared error (MSE) of any estimator can not go below a bound, which is given by

$$E\left\{[\hat{\mathbf{x}}(\mathbf{z})-\mathbf{x}]\,[\hat{\mathbf{x}}(\mathbf{z})-\mathbf{x}]^T\right\} \geq J^{-1} \tag{1}$$

where J is the Fisher information matrix

$$J = E\left\{-\Delta_{\mathbf{x}}^{\mathbf{x}} \log p(\mathbf{x},\mathbf{z})\right\} \tag{2}$$

and the expectation is taken with respect to $p(\mathbf{x},\mathbf{z})$, which is the joint PDF of the pair $(\mathbf{x},\mathbf{z})$. Δ denotes the second-order derivative operator, namely

$$\Delta_{\mathbf{x}}^{\mathbf{y}} = \nabla_{\mathbf{x}} \nabla_{\mathbf{y}}^T \tag{3}$$

in which ∇ denotes the gradient operator. Unbiasedness of the estimator $\hat{\mathbf{x}}$ is not required for the Bayesian CRLB. The mild conditions and proof of this inequality can be found in [2].

The sequential Bayesian estimation problem is to find the estimate of the state from the measurements (observations) over time. The evolution of the state sequence $\mathbf{x}_k$ is assumed to be an unobserved first order Markov process, and is modeled as

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k, \mathbf{u}_k) \tag{4}$$

where $f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is, in general, a nonlinear function of state $\mathbf{x}$, and $\{\mathbf{u}_k, k \in \{0\} \bigcup \mathbb{N}\}$ is an independent white process noise. n_u is the dimension of the noise vector $\mathbf{u}_k$. The PDF of the initial state $\mathbf{x}_0$ is assumed to be known. The observations about the state are obtained from the measurement equation

$$\mathbf{z}_k = h_k(\mathbf{x}_k, \mathbf{v}_k) \tag{5}$$

where $h_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_z}$ is, in general, a nonlinear function, $\{\mathbf{v}_k, k \in \mathbb{N}\}$ is the measurement noise sequence, which is independent of $\mathbf{x}_k$ as well as $\mathbf{u}_k$. n_v is the dimension of the noise vector $\mathbf{v}_k$. Since process noise and measurement noise are assumed to be independent, $\mathbf{x}_{k+1}$ is independent of $\mathbf{z}_{1:k}$ given $\mathbf{x}_k$, which means that $p(\mathbf{x}_{k+1}|\mathbf{x}_k, \mathbf{z}_{1:k}) = p(\mathbf{x}_{k+1}|\mathbf{x}_k)$.

If we denote the states and measurements up to time k as $\mathbf{x}_{0:k}$ and $\mathbf{z}_{1:k}$, then the joint PDF of $(\mathbf{x}_{0:k}, \mathbf{z}_{1:k})$ can be determined from (4) and (5) with known initial PDF $p(\mathbf{x}_0)$ and noise models for $\mathbf{u}_k$ and $\mathbf{v}_k$

$$p(\mathbf{x}_{0:k}, \mathbf{z}_{1:k}) = p(\mathbf{x}_0) \prod_{i=1}^{k} p(\mathbf{x}_i|\mathbf{x}_{i-1}) \prod_{j=1}^{k} p(\mathbf{z}_j|\mathbf{x}_j). \tag{6}$$

If we consider $\mathbf{x}_{0:k}$ as a vector with dimension $(k+1)n_x$, and define $J(\mathbf{x}_{0:k})$ to be the $(k+1)n_x \times (k+1)n_x$ Fisher information matrix of $\mathbf{x}_{0:k}$ derived from the joint PDF $p(\mathbf{x}_{0:k}, \mathbf{z}_{1:k})$, (1) becomes

$$E\left\{[\hat{\mathbf{x}}_{0:k}(\mathbf{z}_{1:k}) - \mathbf{x}_{0:k}][\hat{\mathbf{x}}_{0:k}(\mathbf{z}_{1:k}) - \mathbf{x}_{0:k}]^T\right\} \geq J^{-1}(\mathbf{x}_{0:k}). \tag{7}$$

Let us define J_k as the matrix whose inverse equals the $n_x \times n_x$ lower-right corner submatrix of $J^{-1}(\mathbf{x}_{0:k})$. Then, the MSE of the estimate for $\mathbf{x}_k$ is bounded by J_k^{-1}.

J_k can be obtained directly from the computed inverse of the $(k+1)n_x \times (k+1)n_x$ matrix $J(\mathbf{x}_{0:k})$. However, this is not an efficient approach. In [1], Tichavsky *et al.* provide an elegant recursive approach to calculate J_k without manipulating the large matrices at each time k

$$J_{k+1} = D_k^{22} - D_k^{21}\left(J_k + D_k^{11}\right)^{-1} D_k^{12} \tag{8}$$

where

$$D_k^{11} = E\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \tag{9}$$

$$D_k^{12} = E\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_{k+1}} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} = \left(D_k^{21}\right)^T \tag{10}$$

$$\begin{aligned} D_k^{22} &= E\left\{-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} [\log p(\mathbf{x}_{k+1}|\mathbf{x}_k) + \log p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})]\right\} \\ &= D_k^{22,a} + D_k^{22,b}. \end{aligned} \tag{11}$$

Conventional PCRLB considers the measurements as random vectors, and at any particular time k, the bound is calculated by taking the average of both the measurements and the states up to time k. In many cases, besides the two system equations, some of the measurements are available, for example, the measurements up to time $k-1$, $\mathbf{z}_{1:k-1}$. In this paper, we introduce the notion of conditional PCRLB, which utilizes the information contained in the available measurements. The proposed bound is an online bound, and it gives us more accurate indication on the performance of the estimator at the upcoming time than the conventional PCRLB.

III. Conditional PCRLB for Nonlinear Dynamical Systems

The conditional PCRLB sets a bound on the performance of estimating $\mathbf{x}_{0:k+1}$ when the new measurement $\mathbf{z}_{k+1}$ becomes available given that the past measurements up to time k are all known. Here the measurements up to time k are taken as realizations rather than random vectors.

Definition 1: Conditional estimator $\hat{\mathbf{x}}_{0:k+1}(\mathbf{z}_{k+1}|\mathbf{z}_{1:k})$ is defined as a function of the observed data $\mathbf{z}_{k+1}$ given the existing measurements $\mathbf{z}_{1:k}$.

Definition 2: Mean squared error of the conditional estimator at time $k+1$ is defined as follows:

$$\begin{aligned} \mathrm{MSE}(\hat{\mathbf{x}}_{0:k+1}|\mathbf{z}_{1:k}) &\triangleq E\left\{\tilde{\mathbf{x}}_{0:k+1}\tilde{\mathbf{x}}_{0:k+1}^T|\mathbf{z}_{1:k}\right\} \\ &= \int \tilde{\mathbf{x}}_{0:k+1}\tilde{\mathbf{x}}_{0:k+1}^T p_{k+1}^c d\mathbf{x}_{0:k+1} d\mathbf{z}_{k+1} \end{aligned} \tag{12}$$

where $\tilde{\mathbf{x}}_{0:k+1} \triangleq \hat{\mathbf{x}}_{0:k+1} - \mathbf{x}_{0:k+1}$ is the estimation error, and $p_{k+1}^c \triangleq p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$.

Definition 3: Let $I(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$ be the $(k+2)n_x \times (k+2)n_x$ conditional Fisher information matrix of the state vector $\mathbf{x}_{0:k+1}$ from time 0 to $k+1$:

$$\begin{aligned} I(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k}) &\triangleq E\left\{-\left[\Delta_{\mathbf{x}_{0:k+1}}^{\mathbf{x}_{0:k+1}} \log p_{k+1}^c\right]|\mathbf{z}_{1:k}\right\} \\ &= -\int \left[\Delta_{\mathbf{x}_{0:k+1}}^{\mathbf{x}_{0:k+1}} \log p_{k+1}^c\right] p_{k+1}^c d\mathbf{x}_{0:k+1} d\mathbf{z}_{k+1}. \end{aligned} \tag{13}$$

With the above definitions, we give the conditional posterior CRLB inequality.

Proposition 1: The conditional mean squared error of the state vector $\mathbf{x}_{0:k+1}$ is lower bounded by the inverse of the conditional FIM

$$E\left\{\tilde{\mathbf{x}}_{0:k+1}\tilde{\mathbf{x}}_{0:k+1}^T|\mathbf{z}_{1:k}\right\} \geq I^{-1}(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k}) \tag{14}$$

The proof of Proposition 1 is similar to the one for the unconditional PCRLB presented in [2]. It is not presented here for brevity and is available in [9].

Definition 4: $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is defined as the conditional Fisher information matrix for estimating $\mathbf{x}_{k+1}$, and $L^{-1}(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is equal to the $n_x \times n_x$ lower-right block of $I^{-1}(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$.

By definition, $L^{-1}(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is a bound on the MSE of the estimate for $\mathbf{x}_{k+1}$ given $\mathbf{z}_{1:k}$. At time k, the conditional PCRLB, $L^{-1}(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$, provides a predicted estimator performance limit for the upcoming time $k+1$, given the measurements up to time k. Therefore, it is very useful for the sensor/resource management for target tracking in sensor networks [10], [11]. Here, we propose an iterative approach to calculate $L^{-1}(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ without manipulating the large matrix $I(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$. This iterative approach is facilitated by an auxiliary FIM, which is defined below.

Definition 5: The auxiliary Fisher information matrix for the state vector from time 0 to k is defined as

$$\begin{aligned} I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) &\triangleq E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left\{-\Delta_{\mathbf{x}_{0:k}}^{\mathbf{x}_{0:k}} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right\} \\ &= -\int \left[\Delta_{\mathbf{x}_{0:k}}^{\mathbf{x}_{0:k}} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right] p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) d\mathbf{x}_{0:k}. \end{aligned} \tag{15}$$

Definition 6: We define $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ as the auxiliary Fisher information matrix for $\mathbf{x}_k$, and $L_A^{-1}(\mathbf{x}_k|\mathbf{z}_{1:k})$ is equal to the $n_x \times n_x$ lower-right block of $I_A^{-1}(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$.

The matrix inversion formula [12] is heavily used for deriving the recursive version of the conditional PCRLB. We include it here for completeness

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}B^TA^{-1} & E^{-1} \end{bmatrix} \tag{16}$$

where A, B, and C are submatrices with appropriate dimensions, and $D = A - BC^{-1}B^T$, $E = C - B^TA^{-1}B$.

By definition, the inverse of $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is the lower-right block of $I^{-1}(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$. Instead of calculating $I^{-1}(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$ directly, the following theorem gives a simple approach for computing $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$.

Theorem 1: The sequence of conditional Fisher information $\{L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})\}$ for estimating state vectors $\{\mathbf{x}_{k+1}\}$ can be computed as follows:

$$L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k}) = B_k^{22} - B_k^{21}\left[B_k^{11} + L_A(\mathbf{x}_k|\mathbf{z}_{1:k})\right]^{-1} B_k^{12} \tag{17}$$

where

$$B_k^{11} = E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \tag{18}$$

$$B_k^{12} = E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_{k+1}}\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} = \left(B_k^{21}\right)^T \tag{19}$$

$$B_k^{22} = E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}}\left[\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)+\log p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})\right]\right\}. \tag{20}$$

Proof: The conditional Fisher information matrix can be decomposed as follows:

$$\begin{aligned} I(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k}) &= E_{p_{k+1}^c}(-1)\begin{bmatrix} \Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_{0:k-1}} & \Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_k} & \Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_{k+1}} \\ \Delta_{\mathbf{x}_k}^{\mathbf{x}_{0:k-1}} & \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} & \Delta_{\mathbf{x}_k}^{\mathbf{x}_{k+1}} \\ \Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{0:k-1}} & \Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_k} & \Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} \end{bmatrix}\log p_{k+1}^c \\ &= \begin{bmatrix} A_k^{11} & A_k^{12} & \mathbf{0} \\ A_k^{21} & A_k^{22}+B_k^{11} & B_k^{12} \\ \mathbf{0} & B_k^{21} & B_k^{22} \end{bmatrix} \end{aligned} \tag{21}$$

where

$$\begin{aligned} A_k^{11} &= E_{p_{k+1}^c}\left[-\Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_{0:k-1}}\log p_{k+1}^c\right] \\ &= E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_{0:k-1}}\log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right]. \end{aligned} \tag{22}$$

In a similar manner, A_k^{12} can be derived as

$$\begin{aligned} A_k^{12} &= E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{0:k-1}}^{\mathbf{x}_k}\log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right] \\ &= \left(A_k^{21}\right)^T \end{aligned} \tag{23}$$

$$A_k^{22} + B_k^{11} = E_{p_{k+1}^c}\left[-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p_{k+1}^c\right] \tag{24}$$

where

$$A_k^{22} = E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right] \tag{25}$$

and B_k^{11} has been defined in (18). The conditional Fisher information FIM $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is equal to the inverse of the lower-right submatrix of $I^{-1}(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k})$. So

$$\begin{aligned} L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k}) =& \\ & B_k^{22} - \begin{bmatrix}\mathbf{0} & B_k^{21}\end{bmatrix}\begin{bmatrix} A_k^{11} & A_k^{12} \\ A_k^{21} & A_k^{22}+B_k^{11}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0} \\ B_k^{12}\end{bmatrix} \\ =& B_k^{22} - B_k^{21}\left[B_k^{11}+L_A(\mathbf{x}_k|\mathbf{z}_{1:k})\right]^{-1}B_k^{12} \end{aligned} \tag{26}$$

where

$$L_A(\mathbf{x}_k|\mathbf{z}_{1:k}) = A_k^{22} - A_k^{21}\left(A_k^{11}\right)^{-1}A_k^{12}. \tag{27}$$

The identity in (27) will be provided later in deriving an approximate approach to compute the conditional PCRLB recursively. Q.E.D.

Theorem 1 indicates that the conditional Fisher information at the current time step, $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$, can not be directly calculated from that at the previous time step, $L(\mathbf{x}_k|\mathbf{z}_{1:k-1})$. Instead, its evaluation has to be facilitated by the auxiliary Fisher information $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$. This implies that the heuristically modified conditional PCRLB presented in [7], which has a direct recursion from $L(\mathbf{x}_k|\mathbf{z}_{1:k-1})$ to $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$, does not yield the exact conditional PCRLB as provided in Definitions 3 and 4.

In Theorem 1, a recursive approach is provided to predict the performance of the nonlinear filter at the next time step, based on the measurements up to the current time. Now let us investigate the relationship between the conditional PCRLB presented in Theorem 1 and the unconditional PCRLB with renewal strategy proposed in [3]. In the unconditional PCRLB with renewal strategy, the counterpart of the one-step-ahead conditional PCRLB works as follows. At each time k, the system prior PDF is re-initialized with the posterior PDF $p_0(\mathbf{x}_k) = p(\mathbf{x}_k|\mathbf{z}_{1:k})$. Accordingly, $E\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_k|\mathbf{z}_{1:k})\}$ takes the place of J_k in (8). The Fisher information J_{k+1} at time $k+1$ is then calculated by one-step recursion using (8) through (11), where the expectations are taken with respect to $p(\mathbf{x}_{k:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$.

We summarize the relationship between the one-step ahead conditional PCRLB and the recursive unconditional PCRLB that renews its prior at each time in the following lemma.

Lemma 1: The conditional Fisher information matrix $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ provided in Theorem 1 is different from J_{k+1}, calculated by one-step recursion using (8) through (11) and setting the system state prior PDF $p_0(\mathbf{x}_k)$ as $p(\mathbf{x}_k|\mathbf{z}_{1:k})$, provided that $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ is different from $\widetilde{J}_k$, which is defined as $E_{p(\mathbf{x}_k|\mathbf{z}_{1:k})}\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_k|\mathbf{z}_{1:k})\}$.

Proof: In the recursive unconditional PCRLB that renews its prior at each time, according to (8),

$$J_{k+1} = D_k^{22} - D_k^{21}\left(\widetilde{J}_k + D_k^{11}\right)^{-1}D_k^{12} \tag{28}$$

where

$$\widetilde{J}_k = E_{p(\mathbf{x}_k|\mathbf{z}_{1:k})}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_k|\mathbf{z}_{1:k})\right\}. \tag{29}$$

Based on Theorem 1, the conditional FIM is given as

$$L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k}) = B_k^{22} - B_k^{21}\left[B_k^{11}+L_A(\mathbf{x}_k|\mathbf{z}_{1:k})\right]^{-1}B_k^{12}. \tag{30}$$

Since in the unconditional PCRLB that renews its prior at each time, the expectations are taken with respect to $p(\mathbf{x}_{k:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$. According to (9), it is easy to show that

$$\begin{aligned} D_k^{11} &= E_{p(\mathbf{x}_{k:k+1},\mathbf{z}_{k+1}|\mathbf{z}_{1:k})}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \\ &= E_{p(\mathbf{x}_{0:k+1},\mathbf{z}_{k+1}|\mathbf{z}_{1:k})}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} = B_k^{11}. \end{aligned} \tag{31}$$

Similarly, it can be proved that $B_k^{12} = D_k^{12}$, $B_k^{21} = D_k^{21}$, and $B_k^{22} = D_k^{22}$. Now, the right-hand sides (RHSs) of (28) and (30) differ by only one term, which is either $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ or $\widetilde{J}_k$. Hence, if $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ is different from $\widetilde{J}_k$, in general, the conditional Fisher information matrix $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is different from J_{k+1}, which is calculated using the unconditional PCRLB that renews its prior at each time. Q.E.D.

The auxiliary Fisher information matrix has been defined in a way such that its inverse, $L_A^{-1}(\mathbf{x}_k|\mathbf{z}_{1:k})$, is equal to the $n_x \times n_x$ lower-right block of $[E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\{-\Delta_{\mathbf{x}_{0:k}}^{\mathbf{x}_{0:k}} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\}]^{-1}$. It can be shown that in a linear and Gaussian system, $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ and $\widetilde{J}_k$ are equivalent, so that the conditional PCRLB and the unconditional PCRLB that renews its prior at each time are equivalent. For nonlinear/non-Gaussian systems, the calculation of $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ and $\widetilde{J}_k$ involves complex integrations and analytical results are intractable in general. Hence, direct comparison is very difficult. However, we demonstrate their difference through a simulation for a particular nonlinear system. The results are shown in Experiment V in Section V. From the numerical results, we can see that $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ is not equal to $\widetilde{J}_k = E\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{x}_k|\mathbf{z}_{1:k})\}$. This in turn implies that $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ and J_{k+1} are different in general.

One problem that is left in the proof of Theorem 1 is the inverse of the auxiliary Fisher Information matrix, $L_A^{-1}(\mathbf{x}_k|\mathbf{z}_{1:k})$, which is equal to the $n_x \times n_x$ lower-right block of $I_A^{-1}(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$. Direct computation of $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ involves the inverse of the matrix $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ of size $(k+1)n_x \times (k+1)n_x$. Therefore, we provide a recursive method for computing $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ approximately, which is much more efficient.

Now let us derive the approximate recursive formula to calculate $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$. $I_A(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1})$ can be decomposed as

$$I_A(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1}) = \begin{bmatrix} A_{k-1}^{11} & A_{k-1}^{12} \\ A_{k-1}^{21} & A_{k-1}^{22} \end{bmatrix}. \tag{32}$$

Taking the inverse of the above matrix and applying (16), we have

$$L_A(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1}) = A_{k-1}^{22} - A_{k-1}^{21}\left(A_{k-1}^{11}\right)^{-1} A_{k-1}^{12}. \tag{33}$$

Now consider $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$. We have (34), shown at the bottom of the page, where $\mathbf{0}$s stand for blocks of zeros of appropriate dimensions. In general, there is no recursive method to calculate $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$. This is because the measurement $\mathbf{z}_k$ provides new information about the system state in the past $(\mathbf{x}_{0:k-1})$, which will affect the top-left part of $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$. As we can see, $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ is a block tridiagonal matrix. The top-left submatrix of $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ is a function of $\mathbf{z}_k$, which can be approximated by its expectation with respect to $p(\mathbf{z}_k|\mathbf{z}_{1:k-1})$, if we take $\mathbf{z}_k$ and $\mathbf{z}_{1:k-1}$ as random vector and measurement realizations respectively. So we have

$$\begin{aligned} &E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{0:k-2}} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right] \\ &\quad\approx E_{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})} \\ &\qquad\times \left\{E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{0:k-2}} \log p(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1})\right]\right\} \\ &\quad= E_{p(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1})}\left[-\Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{0:k-2}} \log p(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1})\right] \\ &\quad= A_{k-1}^{11} \end{aligned} \tag{35}$$

where (6) has been used. Because the auxiliary Fisher information matrix $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ is equal to the inverse of the lower-right block of $I_A^{-1}(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$, we have

$$\begin{aligned} &L_A(\mathbf{x}_k|\mathbf{z}_{1:k}) \\ &\quad\approx S_k^{22} - \begin{bmatrix}\mathbf{0} & S_k^{21}\end{bmatrix} \begin{bmatrix} A_{k-1}^{11} & A_{k-1}^{12} \\ A_{k-1}^{21} & A_{k-1}^{22}+S_k^{11} \end{bmatrix}^{-1} \begin{bmatrix}\mathbf{0} \\ S_k^{12}\end{bmatrix} \\ &\quad= S_k^{22} - S_k^{21}\left[S_k^{11} + A_{k-1}^{22} - A_{k-1}^{21}\left(A_{k-1}^{11}\right)^{-1} A_{k-1}^{12}\right]^{-1} S_k^{12} \\ &\quad= S_k^{22} - S_k^{21}\left[S_k^{11} + L_A(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})\right]^{-1} S_k^{12} \end{aligned} \tag{36}$$

where

$$S_k^{11} \triangleq E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{k-1}} \log p(\mathbf{x}_k|\mathbf{x}_{k-1})\right] \tag{37}$$

$$S_k^{12} \triangleq E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \log p(\mathbf{x}_k|\mathbf{x}_{k-1})\right] = \left(S_k^{21}\right)^T \tag{38}$$

$$\begin{aligned} S_k^{22} \triangleq\, &E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})} \\ &\times \left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\left[\log p(\mathbf{x}_k|\mathbf{x}_{k-1}) + \log p(\mathbf{z}_k|\mathbf{x}_k)\right]\right\}. \end{aligned} \tag{39}$$

In summary, the sequence of $\{L_A(\mathbf{x}_k|\mathbf{z}_{1:k})\}$ can be computed recursively as provided in the following approximation:

Approximation 1:

$$L_A(\mathbf{x}_k|\mathbf{z}_{1:k}) \approx S_k^{22} - S_k^{21}\left[S_k^{11} + L_A(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})\right]^{-1} S_k^{12}. \tag{40}$$

In the recursive evaluation approach, the approximation made in (35) may cause cumulative error. The theoretical analysis of the cumulative approximation error is very difficult. In Section V, this approximation method is justified through simulation experiments for a highly nonlinear system. In the experiments, the conditional PCRLB evaluated using Theorem 1 and the method with the approximated $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ provided by Approximation 1 and that evaluated based on Theorem 1 and the exact $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ by calculating (34) without approximation yield results that are very close to each other.

$$\begin{aligned} I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) &= E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left\{(-1)\begin{bmatrix} \Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{0:k-2}} & \Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{k-1}} & \mathbf{0} \\ \Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{0:k-2}} & \Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{k-1}} & \Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_k} \\ \mathbf{0} & \Delta_{\mathbf{x}_k}^{\mathbf{x}_{k-1}} & \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \end{bmatrix} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right\} \\ &= \begin{bmatrix} E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta_{\mathbf{x}_{0:k-2}}^{\mathbf{x}_{0:k-2}} \log p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})\right] & A_{k-1}^{12} & \mathbf{0} \\ A_{k-1}^{21} & A_{k-1}^{22}+S_k^{11} & S_k^{12} \\ \mathbf{0} & S_k^{21} & S_k^{22} \end{bmatrix} \end{aligned} \tag{34}$$

IV. A Sequential Monte Carlo Solution for Conditional PCRLB

In Section III, we have shown that given the available measurement data $\mathbf{z}_{1:k}$, the conditional Fisher information matrix $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ can be recursively calculated according to Theorem 1 and Approximation 1. However, in most cases, direct computation of B_k^{11}, B_k^{12}, B_k^{22}, S_k^{11}, S_k^{12}, and S_k^{22} involves high-dimensional integration, and in general analytical solutions do not exist. Here sequential Monte Carlo methods, or particle filters, are proposed to evaluate these terms. For nonlinear non-Gaussian Bayesian recursive estimation problems, the particle filter is a very popular and powerful tool. Based on importance sampling techniques, particle filters approximate the high-dimension integration using Monte Carlo simulations and interested readers are referred to [13] and [14] for details. For nonlinear dynamic systems that use particle filters for state estimation, the proposed particle filter based conditional PCRLB evaluation solution is very convenient, since the auxiliary Fisher information matrix $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ and the conditional Fisher information matrix $L(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ can be evaluated online as by-products of the particle filter state estimation process, as shown later in the paper.

Under the assumptions that the states evolve according to a first-order Markov process and the observations are conditionally independent given the states, the PDF $p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$ can be factorized as

$$p_{k+1}^c = p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|\mathbf{x}_k)p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}). \tag{41}$$

Letting N denote the number of particles used in the particle filter, the posterior PDF $p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ at time k can be approximated by the particles [14]

$$p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} \delta\left(\mathbf{x}_{0:k} - \mathbf{x}_{0:k}^l\right) \tag{42}$$

where we assume that the resampling has been performed at time k, so that each particle has an identical weight $1/N$. With (41) and (42), we can readily show that

$$p_{k+1}^c \approx \frac{1}{N} p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|\mathbf{x}_k)\sum_{l=1}^{N} \delta\left(\mathbf{x}_{0:k} - \mathbf{x}_{0:k}^l\right).$$

We also derive another approximation for $p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$, which is given by the following proposition.

Proposition 2:

$$p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} \delta\left(\mathbf{x}_{0:k+1} - \mathbf{x}_{0:k+1}^l\right) \times p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right). \tag{43}$$

Proof: See Appendix A.

Note that even though approximations in (42) and (43) require that each particle represents one system state realization from time 0 to time k ($\mathbf{x}_{0:k}$), we will show later that for calculating conditional PCRLB at time step k, it is sufficient for each particle to keep system state realization at time steps $k-1$ and k only, which means that we only need to keep $\mathbf{x}_{k-1:k}$ for computation. This results in a significantly reduced burden for system memory.

In this section, we will consider the general form of the conditional PCRLB for any nonlinear/non-Gaussian dynamic system, as well as two special cases.

A. General Formulation

The general form is given to calculate each component in (17) and (40) for any nonlinear/non-Gaussian system. In the following equations, the superscripts represent the particle index. We also assume that the derivatives and expectations exist and the integration and derivatives are exchangeable.

1) $\mathbf{B_k^{11}}$*:*

$$\begin{aligned} B_k^{11} &= E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k}\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \\ &= E_{p_{k+1}^c}\left[\frac{\nabla_{\mathbf{x}_k} p(\mathbf{x}_{k+1}|\mathbf{x}_k)\nabla_{\mathbf{x}_k}^T p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p^2(\mathbf{x}_{k+1}|\mathbf{x}_k)} - \frac{\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p(\mathbf{x}_{k+1}|\mathbf{x}_k)}\right]. \end{aligned} \tag{44}$$

First, it is easy to show that

$$E_{p_{k+1}^c}\left[\frac{\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p(\mathbf{x}_{k+1}|\mathbf{x}_k)}\right] = 0. \tag{45}$$

Now let us define

$$g_1(\mathbf{x}_k, \mathbf{x}_{k+1}) \triangleq \frac{\nabla_{\mathbf{x}_k} p(\mathbf{x}_{k+1}|\mathbf{x}_k)\nabla_{\mathbf{x}_k}^T p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p^2(\mathbf{x}_{k+1}|\mathbf{x}_k)}. \tag{46}$$

By substituting (43), (45), and (46) into (44), we have

$$B_k^{11} = E_{p_{k+1}^c}\left[g_1(\mathbf{x}_k, \mathbf{x}_{k+1})\right] \approx \frac{1}{N}\sum_{l=1}^{N} g_1\left(\mathbf{x}_k^l, \mathbf{x}_{k+1}^l\right). \tag{47}$$

2) $\mathbf{B_k^{12}}$*:* Following a similar procedure, we have

$$B_k^{12} \approx \frac{1}{N} \times \sum_{l=1}^{N} \left.\frac{\nabla_{\mathbf{x}_k} p(\mathbf{x}_{k+1}|\mathbf{x}_k)\nabla_{\mathbf{x}_{k+1}}^T p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p^2(\mathbf{x}_{k+1}|\mathbf{x}_k)}\right|_{\{\mathbf{x}_k,\mathbf{x}_{k+1}\}=\left\{\mathbf{x}_k^l,\mathbf{x}_{k+1}^l\right\}}. \tag{48}$$

3) $\mathbf{B_k^{22} = B_k^{22,a} + B_k^{22,b}}$*:*

$$B_k^{22,a} \approx \frac{1}{N} \times \sum_{l=1}^{N} \left.\frac{\nabla_{\mathbf{x}_{k+1}} p(\mathbf{x}_{k+1}|\mathbf{x}_k)\nabla_{\mathbf{x}_{k+1}}^T p(\mathbf{x}_{k+1}|\mathbf{x}_k)}{p^2(\mathbf{x}_{k+1}|\mathbf{x}_k)}\right|_{\{\mathbf{x}_k,\mathbf{x}_{k+1}\}=\left\{\mathbf{x}_k^l,\mathbf{x}_{k+1}^l\right\}} \tag{49}$$

$$\begin{aligned} B_k^{22,b} &= E_{p_{k+1}^c}\left[-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}}\log p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})\right] \\ &\approx \frac{1}{N}\sum_{l=1}^{N} g_2\left(\mathbf{x}_{k+1}^l\right) \end{aligned} \tag{50}$$

where

$$g_2(\mathbf{x}_{k+1}) \triangleq \int \frac{\nabla_{\mathbf{x}_{k+1}} p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}) \nabla^T_{\mathbf{x}_{k+1}} p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})}{p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})} d\mathbf{z}_{k+1}. \quad (51)$$

For the cases where the integration in (51) does not have a closed-form solution, it can be approximated by numerical integration approaches.

4) $\mathbf{S_k^{11}}$*:*

$$\begin{aligned} S_k^{11} &= E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[-\Delta^{\mathbf{x}_{k-1}}_{\mathbf{x}_{k-1}} \log p(\mathbf{x}_k|\mathbf{x}_{k-1})\right] \\ &= E_{p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})}\left[\frac{\nabla_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1}) \nabla^T_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p^2(\mathbf{x}_k|\mathbf{x}_{k-1})} \right. \\ &\qquad \left. - \frac{\Delta^{\mathbf{x}_{k-1}}_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p(\mathbf{x}_k|\mathbf{x}_{k-1})}\right]. \end{aligned} \quad (52)$$

Since $\mathbf{z}_{1:k}$ are available measurement data, the posterior PDF $p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ can be approximated through sequential Monte Carlo approaches. Plugging (42) into the above equation, we have

$$S_k^{11} \approx \frac{1}{N} \sum_{l=1}^{N} g_3\left(\mathbf{x}^l_{k-1}, \mathbf{x}^l_k\right) \quad (53)$$

where

$$\begin{aligned} g_3(\mathbf{x}_{k-1}, \mathbf{x}_k) \triangleq{} & \frac{\nabla_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1}) \nabla^T_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p^2(\mathbf{x}_k|\mathbf{x}_{k-1})} \\ & - \frac{\Delta^{\mathbf{x}_{k-1}}_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p(\mathbf{x}_k|\mathbf{x}_{k-1})}. \end{aligned} \quad (54)$$

5) $\mathbf{S_k^{12}}$*:* Following a similar procedure as in calculating S_k^{11}, we have

$$S_k^{12} \approx \frac{1}{N} \sum_{l=1}^{N} g_4\left(\mathbf{x}^l_{k-1}, \mathbf{x}^l_k\right) \quad (55)$$

where

$$\begin{aligned} g_4(\mathbf{x}_{k-1}, \mathbf{x}_k) \triangleq{} & \frac{\nabla_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1}) \nabla^T_{\mathbf{x}_k} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p^2(\mathbf{x}_k|\mathbf{x}_{k-1})} \\ & - \frac{\Delta^{\mathbf{x}_k}_{\mathbf{x}_{k-1}} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p(\mathbf{x}_k|\mathbf{x}_{k-1})}. \end{aligned} \quad (56)$$

6) $\mathbf{S_k^{22}}$*:* S_k^{22} consists of two parts, $S_k^{22} = S_k^{22,a} + S_k^{22,b}$, where

$$\begin{aligned} S_k^{22,a} \approx \frac{1}{N} \sum_{l=1}^{N} & \left[\frac{\nabla_{\mathbf{x}_k} p(\mathbf{x}_k|\mathbf{x}_{k-1}) \nabla^T_{\mathbf{x}_k} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p^2(\mathbf{x}_k|\mathbf{x}_{k-1})} \right. \\ & \left.\left. - \frac{\Delta^{\mathbf{x}_k}_{\mathbf{x}_k} p(\mathbf{x}_k|\mathbf{x}_{k-1})}{p(\mathbf{x}_k|\mathbf{x}_{k-1})}\right]\right|_{\{\mathbf{x}_{k-1}, \mathbf{x}_k\} = \{\mathbf{x}^l_{k-1}, \mathbf{x}^l_k\}} \end{aligned} \quad (57)$$

and

$$\begin{aligned} S_k^{22,b} \approx \frac{1}{N} \sum_{l=1}^{N} & \left[\frac{\nabla_{\mathbf{x}_k} p(\mathbf{z}_k|\mathbf{x}_k) \nabla^T_{\mathbf{x}_k} p(\mathbf{z}_k|\mathbf{x}_k)}{p^2(\mathbf{z}_k|\mathbf{x}_k)} \right. \\ & \left.\left. - \frac{\Delta^{\mathbf{x}_k}_{\mathbf{x}_k} p(\mathbf{z}_k|\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{x}_k)}\right]\right|_{\mathbf{x}_k = \mathbf{x}^l_k}. \end{aligned} \quad (58)$$

Taking a closer look at approximations made in this subsection, it is clear that at time step k, for the calculation of the conditional PCRLB at time $k+1$, only the values of system states from time $k-1$ to time $k+1$ ($\mathbf{x}^l_{k-1:k+1}$) are needed. Moreover, when the system transits from step k to step $k+1$, it is sufficient to propagate and update the particle set from $\{\mathbf{x}^l_{k-1:k}\}$ to $\{\mathbf{x}^l_{k:k+1}\}$, where $l = 1, \cdots, N$.

With numerical integrations provided by the particle filter, the approach for evaluating the conditional PCRLB works recursively as follows. At time k, when the measurement $\mathbf{z}_k$ is available, the weights of the particle sets $\{\mathbf{x}^l_{k-1:k}\}$ are updated, which is followed by a re-sampling procedure. Then, each particle $\mathbf{x}^l_{k-1:k}$ has an equal constant weight of $1/N$. $\{\mathbf{x}^l_{k-1:k}\}$ will be used for the calculation of S_k^{11}, S_k^{12}, and S_k^{22}. Then, only the particles $\{\mathbf{x}^l_k\}$ are propagated to the next time step according to (4). The particle set $\{\mathbf{x}^l_{k:k+1}\}$ is used to evaluate B_k^{11}, B_k^{12}, and B_k^{22}. At the end of the kth time step, for the lth particle, only $\mathbf{x}^l_{k:k+1}$ will be preserved and passed to the next $(k+1)$ time step.

Note that the particle filter is an approximate solution to the optimal nonlinear estimator. At the $(k-1)$th iteration, based on the information state $p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})$, which is a function of $\mathbf{z}_{1:k-1}$ and completely summarizes the past of the system in a probabilistic sense [15], the optimal nonlinear estimator calculates the new information state $p(\mathbf{x}_k|\mathbf{z}_{1:k})$ by incorporating the new measurement $\mathbf{z}_k$. As a result, the optimal nonlinear estimator of $\mathbf{x}_k$ at time k is a function of all the measurements up to time k, namely $\mathbf{z}_{1:k}$. The particle filter is nothing but a numerical approximation to the optimal estimator, which recursively updates the particle weights using arriving new measurements, and hence is a function of the measurements up to the current time. Therefore, the conditional PCRLB approximated by sequential Monte Carlo methods depends on the history of the measurements. Details of the optimal estimator and the information state, and the particle filter can be found in [15] and [14] respectively.

Now let us investigate the computational complexities of the recursive conditional FIM, which can be evaluated using Theorem 1 and Approximation 1, and the recursive unconditional PCRLB that renews its prior at each iteration. Lemma 1 shows that these two methods differ only in the computation of $\widetilde{J}_k$ and $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$. When applying the particle filter, at each time k, the complexity for computing the common terms (B_k^{11}, B_k^{12}, and B_k^{22}) in Lemma 1 is linear in the number of particles (N). The terms used in Approximation 1 to recursively compute $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$ (S_k^{11}, S_k^{12}, and S_k^{22}) also have complexities that are linear in N. Since $p(\mathbf{x}_k|\mathbf{z}_{1:k})$ has been represented by a set of particles and associated weights, $\widetilde{J}_k = E_{p(\mathbf{x}_k|\mathbf{z}_{1:k})}\{-\Delta^{\mathbf{x}_k}_{\mathbf{x}_k} \log p(\mathbf{x}_k|\mathbf{z}_{1:k})\}$ could only be evaluated numerically, with a complexity at least linear in N. Thus, the computation of $\widetilde{J}_k$ has a complexity that is at least in the same order of that of $L_A(\mathbf{x}_k|\mathbf{z}_{1:k})$.

B. Additive Gaussian Noise Case

Here, we consider a special case of nonlinear dynamic systems with additive Gaussian noises. It is assumed that the dynamic system has the following state and measurement equations:

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k) + \mathbf{u}_k \quad (59)$$

$$\mathbf{z}_k = h_k(\mathbf{x}_k) + \mathbf{v}_k \quad (60)$$

where $f_k(\cdot)$ and $h_k(\cdot)$ are nonlinear state transition and measurement functions respectively, $\mathbf{u}_k$ is the white Gaussian state process noise with zero mean and covariance matrix Q_k, and $\mathbf{v}_k$ is the white Gaussian measurement noise with zero mean and covariance matrix R_k. The sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ are mutually independent. With these assumptions and notations, the transition prior of the state can be written as

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \frac{1}{(2\pi)^{\frac{n_x}{2}}|Q_k|^{\frac{1}{2}}} \times \exp\left\{-\frac{1}{2}\left[\mathbf{x}_{k+1} - f_k(\mathbf{x}_k)\right]^T Q_k^{-1}\left[\mathbf{x}_{k+1} - f_k(\mathbf{x}_k)\right]\right\}. \quad (61)$$

Taking the logarithm of the above PDF, we have

$$-\log p(\mathbf{x}_{k+1}|\mathbf{x}_k) = c_0 + \frac{1}{2}\left[\mathbf{x}_{k+1} - f_k(\mathbf{x}_k)\right]^T Q_k^{-1} \times \left[\mathbf{x}_{k+1} - f_k(\mathbf{x}_k)\right] \quad (62)$$

where c_0 denotes a constant independent of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$. For vector-valued functions $f(\cdot) = [f_1, f_2, \cdots, f_{n_x}]^T$, the first order and second order derivatives of $f(\cdot)$ are defined respectively as

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \left[\nabla_{\mathbf{x}} f_1, \nabla_{\mathbf{x}} f_2, \cdots, \nabla_{\mathbf{x}} f_{n_x}\right]_{n_x \times n_x} \quad (63)$$

$$\Delta_{\mathbf{x}}^{\mathbf{x}} f(\mathbf{x}) = \left[\Delta_{\mathbf{x}}^{\mathbf{x}} f_1, \Delta_{\mathbf{x}}^{\mathbf{x}} f_2, \cdots, \Delta_{\mathbf{x}}^{\mathbf{x}} f_{n_x}\right]_{n_x \times n_x^2} \quad (64)$$

Therefore, the partial derivatives of $\log p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ with respect to $\mathbf{x}_k$ are

$$\nabla_{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \left[\nabla_{\mathbf{x}_k} f_k(\mathbf{x}_k)\right] Q_k^{-1}\left(\mathbf{x}_{k+1} - f_k(\mathbf{x}_k)\right) \quad (65)$$

and

$$-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \left[\nabla_{\mathbf{x}_k} f_k(\mathbf{x}_k)\right] Q_k^{-1}\left[\nabla_{\mathbf{x}_k}^T f_k(\mathbf{x}_k)\right] - \left[\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} f_k(\mathbf{x}_k)\right] \widetilde{\Sigma}_{\mathbf{u}_k}^{-1} \Upsilon_k^{11} \quad (66)$$

where

$$\widetilde{\Sigma}_{\mathbf{u}_k}^{-1} = \begin{bmatrix} Q_k^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Q_k^{-1} & \mathbf{0} & \vdots \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & Q_k^{-1} \end{bmatrix}_{n_x^2 \times n_x^2} \quad (67)$$

and

$$\Upsilon_k^{11} = \begin{bmatrix} \mathbf{x}_{k+1} - f_k(\mathbf{x}_k) & \dots & \mathbf{0} \\ \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_{k+1} - f_k(\mathbf{x}_k) \end{bmatrix}_{n_x^2 \times n_x}. \quad (68)$$

By substituting (43) and (66) into (44), we have

$$B_k^{11} = E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \approx \frac{1}{N}\sum_{l=1}^{N}\left(\left[\nabla_{\mathbf{x}_k} f(\mathbf{x}_k)\right] Q_k^{-1}\left[\nabla_{\mathbf{x}_k}^T f(\mathbf{x}_k)\right]\right)\Big|_{\mathbf{x}_k=\mathbf{x}_k^l} \quad (69)$$

where the following identity has been used

$$E_{p(\mathbf{x}_{k+1}|\mathbf{x}_k)}\left\{\Upsilon_k^{11}\right\} = \mathbf{0}. \quad (70)$$

From (62), we have

$$-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k) = -Q_k^{-1}\nabla_{\mathbf{x}_k}^T f_k(\mathbf{x}_k).$$

Similarly, we have

$$B_k^{21} = E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_k} \log p(\mathbf{x}_{k+1}|\mathbf{x}_k)\right\} \approx -\frac{1}{N}\sum_{l=1}^{N}\left(Q_k^{-1}\nabla_{\mathbf{x}_k}^T f_k(\mathbf{x}_k)\right)\Big|_{\mathbf{x}_k=\mathbf{x}_k^l}. \quad (71)$$

As for $B_k^{22,a}$ and $B_k^{22,b}$, we have

$$B_k^{22,a} = Q_k^{-1} \quad (72)$$

$$B_k^{22,b} = \frac{1}{N}\sum_{l=1}^{N}\left(\left[\nabla_{\mathbf{x}_{k+1}} h(\mathbf{x}_{k+1})\right] R_{k+1}^{-1} \times \left[\nabla_{\mathbf{x}_{k+1}}^T h(\mathbf{x}_{k+1})\right]\right)\Big|_{\mathbf{x}_{k+1}=\mathbf{x}_{k+1}^l} \quad (73)$$

whose derivation is provided in Appendix B.

The approximations for S_k^{11}, S_k^{21}, and S_k^{22} can be derived similarly. Using (42), we have

$$S_k^{11} \approx \frac{1}{N}\sum_{l=1}^{N} g_5\left(\mathbf{x}_{k-1}^l, \mathbf{x}_k^l\right) \quad (74)$$

where

$$g_5(\mathbf{x}_{k-1}, \mathbf{x}_k) = \left[\nabla_{\mathbf{x}_{k-1}} f_{k-1}(\mathbf{x}_{k-1})\right] Q_{k-1}^{-1}\left[\nabla_{\mathbf{x}_{k-1}}^T f_{k-1}(\mathbf{x}_{k-1})\right] - \left[\Delta_{\mathbf{x}_{k-1}}^{\mathbf{x}_{k-1}} f_{k-1}(\mathbf{x}_{k-1})\right] \widetilde{\Sigma}_{\mathbf{u}_{k-1}}^{-1} \Upsilon_{k-1}^{11}(\mathbf{x}_{k-1}, \mathbf{x}_k) \quad (75)$$

$$S_k^{21} \approx -\frac{1}{N}\sum_{l=1}^{N}\left[Q_{k-1}^{-1}\nabla_{\mathbf{x}_{k-1}}^T f_{k-1}(\mathbf{x}_{k-1})\right]\Big|_{\mathbf{x}_{k-1}=\mathbf{x}_{k-1}^l} \quad (76)$$

$$S_k^{22,a} = Q_{k-1}^{-1} \quad (77)$$

and

$$S_k^{22,b} \approx \frac{1}{N}\sum_{l=1}^{N}\left\{\left[\nabla_{\mathbf{x}_k} h_k(\mathbf{x}_k)\right] R_k^{-1}\left[\nabla_{\mathbf{x}_k}^T h_k(\mathbf{x}_k)\right] - \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} h_k(\mathbf{x}_k) \widetilde{\Sigma}_{\mathbf{v}_k}^{-1} \Upsilon_k^{22,b}\right\}\Big|_{\mathbf{x}_k=\mathbf{x}_k^l} \quad (78)$$

where $\widetilde{\Sigma}_{\mathbf{v}_k}^{-1}$ and $\Upsilon_k^{22,b}$ are defined in Appendix B.

Let us now consider the special case of linear system with additive Gaussian noise. For a linear and Gaussian system, if the initial conditions for both the conditional PCRLB and the unconditional PCRLB are the same, namely

$$J_0 = L_A(\mathbf{x}_0). \quad (79)$$

It is not difficult to show that the conditional PCRLB and PCRLB are equivalent for linear Gaussian dynamic systems, and all the three Fisher information matrices, namely the unconditional Fisher information, the conditional Fisher information and the auxiliary Fisher information, are equivalent. Mathematically,

$$J_k = L_A(\mathbf{x}_k|\mathbf{z}_{1:k}) = L(\mathbf{x}_k|\mathbf{z}_{1:k-1}). \tag{80}$$

The proof of (80) is available in [9], which is skipped here for brevity. Therefore, in a linear/Gaussian system, there is no need to use an online conditional PCRLB bound, which is equivalent to the unconditional PCRLB. Note that in such a case, the Kalman filter is the optimal estimator, where the recursive calculations of filter gains and covariance matrices can be performed offline, since they are independent of the state [15]. In addition, (80) provides the insight that the recursive approximation method to compute conditional PCRLB given by Theorem 1 and Approximation 1 yields the exact result when the system is linear and Gaussian. Therefore, one can expect that for a system with weak nonlinearity and non-Gaussianity, the approximation error incurred by the recursive conditional PCRLB evaluation approach provided by Theorem 1 and Approximation 1 will be smaller than that in a highly nonlinear system.

V. Simulation Results

In this section, we present some illustrative examples to demonstrate the accuracy of the computed bounds. Here we consider the univariate nonstationary growth model (UNGM), a highly nonlinear and bimodal model. The UNGM is very useful in econometrics, and has been used in [13], [16], and [17]. In a UNGM, the dynamic state space equations are given by

$$x_{k+1} = \alpha x_k + \beta \frac{x_k}{1+x_k^2} + \gamma cos(1.2k) + u_k \tag{81}$$

$$z_k = \kappa x_k^2 + v_k \tag{82}$$

where u_k and v_k are the state process noise and measurement noise respectively, and they are white Gaussian with zero means and variances σ_u^2 and σ_v^2.

In the simulations, the conditional MSE is obtained recursively as follows. At time k, the posterior PDF is calculated using a particle filter given the measurement $z_{1:k}$. 1000 Monte Carlo trials are performed to generate independent realizations of z_{k+1} according to the measurement (82). The conditional MSE, $\mathrm{MSE}(\hat{x}_{k+1}|z_{1:k})$, is obtained based on the 1000 Monte Carlo trials. At the next time step $(k+1)$, a single realization of z_{k+1} is picked randomly among the 1000 realizations, and concatenated with the past measurement history to form $z_{1:k+1}$. The particles and weights corresponding to this particular z_{k+1} are stored and used for the $(k+1)$th iteration. The recursive conditional PCRLB with approximations mentioned in Theorem 1 and Approximation 1 is used throughout the experiments, unless otherwise specified. The same particle filter can be used to evaluate both the conditional MSE and the conditional PCRLB.

A. Experiment I

We set parameters $\alpha = 1$, $\beta = 5$, $\gamma = 8$, $\sigma_u^2 = 1$, $\sigma_v^2 = 1$, and $\kappa = 1/20$ for UNGM. Fig. 1 shows the system states and

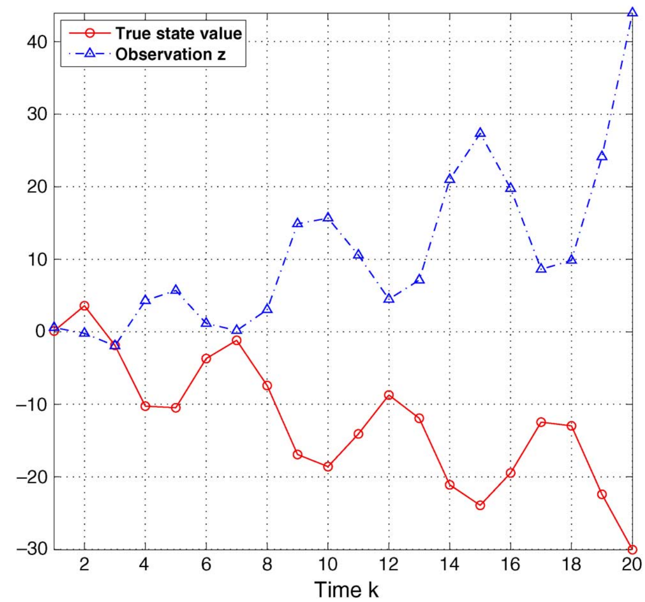


Fig. 1. Plot of the true state x_k and observations z_k.

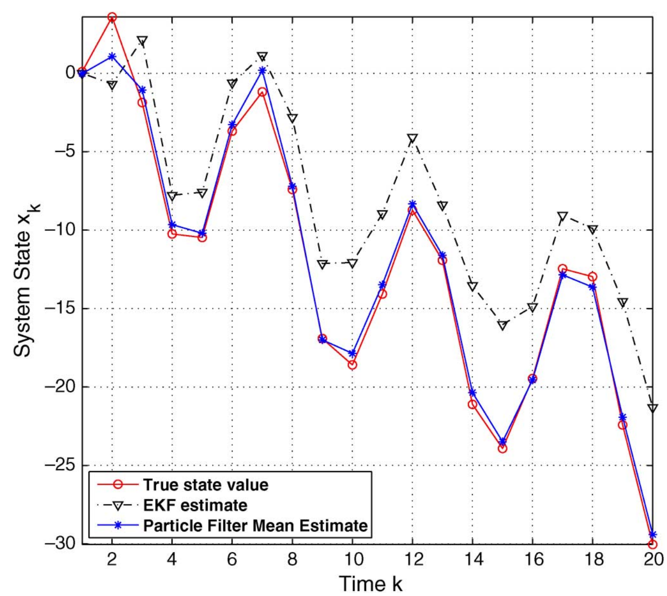


Fig. 2. Plot of filtering results by extended Kalman filter and by particle filter for Example I.

measurements over a period of 20 discrete time steps. Due to the measurement equation of the UNGM specified in (82), there is bi-modality inherent in the filtering problem. This is clearly demonstrated in Fig. 1, where the observation is a quadratic function of the system state, which is then corrupted by an additive noise. In such a case, it is very difficult to track the state using conventional methods, and the particle filter demonstrates better tracking performance than the extended Kalman filter, as illustrated in Fig. 2.

Fig. 3 shows the conditional posterior CRLB and the conditional MSE. It is clearly shown that the conditional PCRLB gives a lower bound on the conditional MSE that an estimator

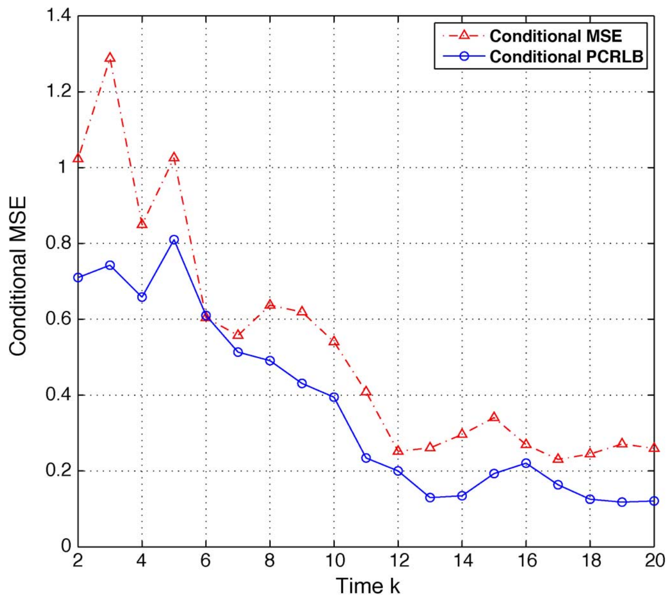


Fig. 3. Plot of conditional posterior CRLB and conditional MSE for Example I.

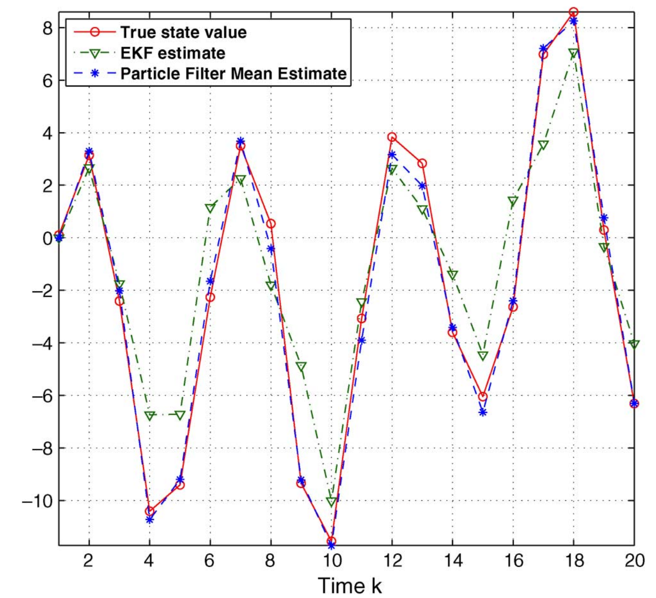


Fig. 4. Plot of filtering results by Extended Kalman filter and by particle filter for Example II.

can achieve. It is also clear that the conditional PCRLB and the conditional MSE follow the same trend.

B. Experiment II

In Section V-A, the choice of parameters for the UNGM makes it highly nonlinear, so that the MSE of the particle filter does not converge to the conditional PCRLB. In Experiment II, we set $\beta = 0.1$, implying a much smaller nonlinear component in the state equation, and set the measurement noise variance as $\sigma_v^2 = 0.01$, meaning a much higher signal to noise ratio (SNR) for the observation. We keep other parameters the same as in Experiment I. In such a case, the UNGM is weakly nonlinear. As illustrated in Fig. 4, the EKF achieves a much better tracking performance than in Experiment I, but the particle filter still outperforms the EKF due to the nonlinearity inherent in this problem. The conditional PCRLB and MSE in Experiment II are shown in Fig. 5. As we can see, the gap between the conditional MSE and the conditional PCRLB is much smaller than that in Experiment I.

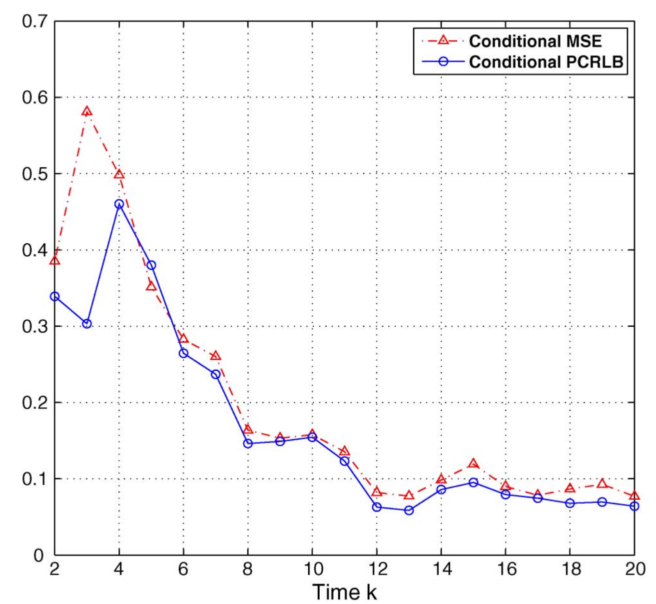


Fig. 5. Plot of conditional posterior CRLB and conditional MSE for Example II.

C. Experiment III

In this experiment, we set the parameters in the UNGM the same as those in Experiment I, and compare the conditional and unconditional PCRLBs in Fig. 6. The conditional PCRLB and conditional MSE are drawn based on a particular realization of the measurement $\mathbf{z}_{1:k}$, and the unconditional PCRLB is obtained by taking the expectation with respect to both the measurements $\mathbf{z}_{1:k}$ and states $\mathbf{x}_{0:k}$. It can be seen that the conditional PCRLB is much tighter than the unconditional PCRLB for the conditional MSE, and it follows the trends of the conditional MSE more faithfully, since the proposed bound utilizes the available measurement information. As a result, the conditional PCRLB can be used as a criterion for managing sensors dynamically for the next time step so that its value is minimized.

D. Experiment IV

In order to recursively calculate the conditional PCRLB, the top-left submatrix of $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{0:k})$ is replaced by its expectation in (34). This might cause propagation of errors due to approximation. Since it is very difficult to analyze the cumulative error theoretically, an experiment is designed to illustrate the approximation errors. In this experiment, the parameters in the UNGM are the same as those in Experiment I. In Fig. 7, the approximate conditional PCRLB evaluated based on Theorem 1 and the approximate recursive method provided by Approximation 1 is compared to the exact conditional PCRLB evaluated

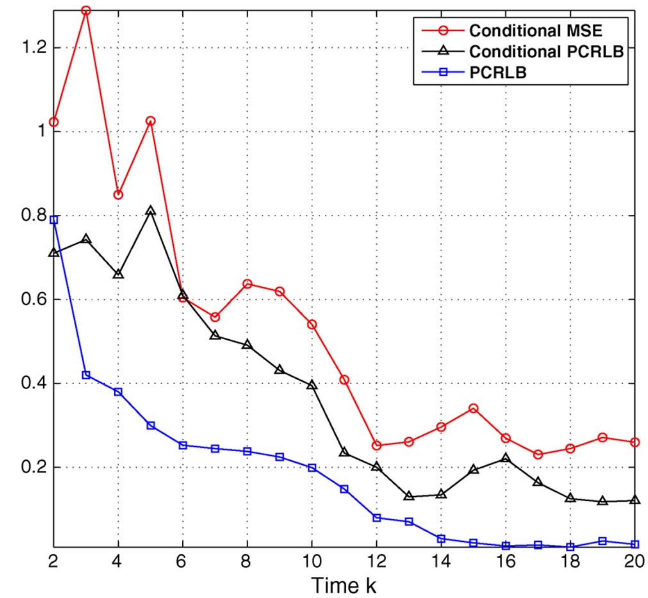


Fig. 6. Comparison of conditional and unconditional PCRLBs for Example III.

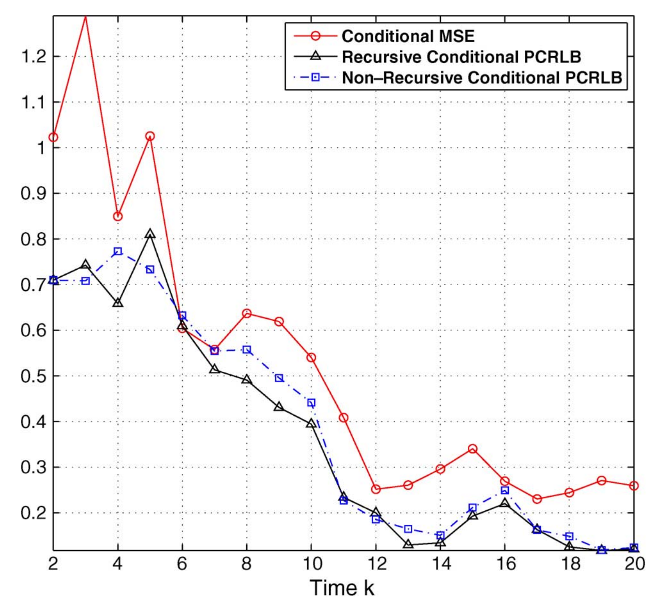


Fig. 7. Comparison of conditional PCRLB between the one with error propagation and the one without error propagation.

using Theorem 1 alone. In the evaluation of the exact conditional PCRLB, by using particle filters, we calculate the complete matrix $I_A(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$ first, then $L_A^{-1}(\mathbf{x}_k|\mathbf{z}_{1:k})$ can be obtained from the lower-right submatrix of $I_A^{-1}(\mathbf{x}_{0:k}|\mathbf{z}_{1:k})$. It is clear from Fig. 7 that the error propagation is not severe even for the highly nonlinear filtering problem. Further, as time increases, the difference between the exact conditional PCRLB and its recursive approximation is getting smaller. Note that the recursive approach requires much less computational effort.

E. Experiment V

To show the difference between the conditional PCRLB and the unconditional PCRLB with renewal strategy, we choose the following system equations in the numerical example

$$\begin{aligned} x_{k+1} &= x_k^2 + u_k \\ z_k &= x_k + v_k \end{aligned} \tag{83}$$

where x_k and z_k are both scalars, and x_0, u_k, and v_k are independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and unit variance. From Lemma 1, we know that the conditional PCRLB $L(x_{k+1}|z_{1:k})$ and the PCRLB with renewal strategy J_{k+1} are different if and only if $L_A(x_k|z_{1:k})$ and $\widetilde{J}_k$ are different. For simplicity, we only consider the case of $k = 1$ in the experiment. According to Definition 5, we have

$$\begin{aligned} I_A(x_{0:1}|z_1) &= E_{p(x_{0:1}|z_1)}\left[-\Delta_{x_{0:1}}^{x_{0:1}} \log p(x_{0:1}|z_1)\right] \\ &= \begin{bmatrix} a & b \\ b & c \end{bmatrix} \end{aligned} \tag{84}$$

With the model used in this experiment and according to Definition 6, the auxiliary FIM $L_A(x_1|z_1) = c - b^2/a$, where

$$\begin{aligned} a &= 1 - 2E_{p(x_{0:1}|z_1)}\{x_1\} + 6E_{p(x_{0:1}|z_1)}\left\{x_0^2\right\} \\ b &= -2E_{p(x_{0:1}|z_1)}\{x_0\} \\ c &= 2. \end{aligned} \tag{85}$$

The evaluation of $L_A(x_1|z_1)$ can be obtained with the help of particle filters. For the unconditional PCRLB with renewal strategy, at $k = 1$ after the re-initialization,

$$\begin{aligned} \widetilde{J}_1 &= E_{p(x_1|z_1)}\left[-\Delta_{x_1}^{x_1} \log p(x_1|z_1)\right] \\ &= E_{p(x_1|z_1)}\left[-\Delta_{x_1}^{x_1} \log p(z_1|x_1)p(x_1)\right] \\ &= 1 + E_{p(x_1|z_1)}\left\{-\Delta_{x_1}^{x_1} \log p(x_1)\right\}. \end{aligned} \tag{86}$$

Given the system (83), the PDF of x_1 can be derived

$$\begin{aligned} p(x_1) &= \frac{1}{2\pi} e^{-\frac{x_1^2}{2}} \int_0^\infty t^{-\frac{1}{2}} e^{-\frac{t^2}{2} + \left(x_1 - \frac{1}{2}\right)t} dt \\ &= \frac{1}{\pi} e^{-\frac{x_1^2}{2}} g(x_1) \end{aligned} \tag{87}$$

where due to the change of variable,

$$g(x_1) \triangleq \int_0^\infty e^{-\frac{t^4}{2} + \left(x_1 - \frac{1}{2}\right)t^2} dt. \tag{88}$$

Finally, we have

$$\Delta_{x_1}^{x_1} \log p(x_1) = -1 + \frac{\Delta_{x_1}^{x_1} g(x_1)}{g(x_1)} - \left[\frac{\nabla_{x_1} g(x_1)}{g(x_1)}\right]^2. \tag{89}$$

$p(x_1|z_1)$ and $p(x_{0:1}|z_1)$ are posterior PDFs, which can be calculated from the particle filter. So given a particular measurement z_1, the value of $L_A(x_1|z_1)$ and $\widetilde{J}_1$ through numerical simulation can be obtained. The simulation results are shown in Table I. Given a variety of measurements z_1's, it is clear that $L_A(x_1|z_1)$ have different values from $\widetilde{J}_1$. It can also be seen that $L_A(x_1|z_1)$ is greater than $\widetilde{J}_1$, which indicates that in this particular case the conditional PCRLB is lower than the PCRLB that renews the prior at each time.

TABLE I
COMPARISON BETWEEN $L_A(x_1|z_1)$ AND $\widetilde{J}_1$

z_1	-1.1414	2.3827	-0.0536	1.3337	-0.4035	0.9550	-0.7795	0.5070	1.2737	-1.9947
$L_A(x_1\|z_1)$	1.9988	1.9594	1.9955	1.9998	1.9989	1.9977	1.9794	1.9763	1.9972	1.9955
$\widetilde{J}_1$	1.8436	1.3275	1.7662	1.5069	1.7493	1.5863	1.8203	1.6576	1.5560	1.8769

VI. CONCLUSION

In this paper, a new notion of conditional PCRLB was proposed, which is conditioned on the actual past measurement realizations and is, therefore, more suitable for online adaptive sensor management. The exact conditional PCRLB and its approximate recursive evaluation formula have been theoretically derived. Further, the sequential Monte Carlo approximation for this bound was proposed to provide a convenient numerical evaluation solution, as a by-product of the particle filtering process. The conditional PCRLB has been compared to existing measurement dependent PCRLBs and shown to be different from them. Simulation results have been provided to demonstrate the effectiveness of the conditional PCRLB in providing online estimation performance prediction, as opposed to the unconditional PCRLB.

Future work will focus on investigating the properties of the proposed bound. Theorem 1 shows that the conditional PCRLB is not only a bound on the filtering estimator $\hat{\mathbf{x}}_{k+1}$, it also sets a bound on the smoothing estimator $\hat{\mathbf{x}}_{0:k}$, when the new measurement $\mathbf{z}_{k+1}$ becomes available. The conditional PCRLB derived in this paper provides an approach to recursively predict the MSE one-step ahead. It can be extended to multi-step ahead cases in the future.

The applications of the proposed bound will be numerous. One possible application area is sensor management in sensor networks. Choosing the most informative set of sensors will improve the tracking performance, while at the same time reduce the requirement for communication bandwidth and the energy needed by sensors for sensing, local computation and communication.

APPENDIX A
PROOF OF PROPOSITION 2

First, the PDF $p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k})$ can be factorized as

$$p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k}) = p(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k+1})p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k}). \quad \text{(A1)}$$

At time k, the prediction or prior $p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ can be approximated as follows. First, a re-sampling procedure is performed, after which each particle has an identical weight, and the posterior PDF is approximated by

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} \delta\left(\mathbf{x}_k - \mathbf{x}_k^l\right). \quad \text{(A2)}$$

The prediction $p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})$ is derived by propagating the particle set $\{\mathbf{x}_k^l, \omega_k^l\}$ from time k to time $k+1$ according to the system model (4)

$$p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} \delta\left(\mathbf{x}_{k+1} - \mathbf{x}_{k+1}^l\right). \quad \text{(A3)}$$

If the transition density of the state $p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ is chosen as the importance density function [13], then the weights at time $k+1$ are given by

$$\omega_{k+1}^l \propto \omega_k^l \; p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right). \quad \text{(A4)}$$

Since re-sampling has been taken at time k, we have $\omega_k^l = 1/N, \forall l$. This yields

$$\omega_{k+1}^l \propto p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right). \quad \text{(A5)}$$

More specifically, the normalized weights are

$$\omega_{k+1}^l = \frac{p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right)}{\sum_{l=1}^{N} p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right).} \quad \text{(A6)}$$

Then, the posterior PDF at time $k+1$ can be approximated by

$$p(\mathbf{x}_{0:k+1}|\mathbf{z}_{1:k+1}) \approx \sum_{l=1}^{N} \omega_{k+1}^l \delta\left(\mathbf{x}_{0:k+1} - \mathbf{x}_{0:k+1}^l\right). \quad \text{(A7)}$$

The second PDF in (A1) involves an integral

$$p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k}) = \int p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|\mathbf{z}_{1:k})d\mathbf{x}_{k+1}. \quad \text{(A8)}$$

Substitution of (A3) into (A8) yields

$$p(\mathbf{z}_{k+1}|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right). \quad \text{(A9)}$$

Further substituting (A6), (A7), and (A9) into (A1) yields

$$p(\mathbf{x}_{0:k+1}, \mathbf{z}_{k+1}|\mathbf{z}_{1:k}) \approx \frac{1}{N}\sum_{l=1}^{N} \delta\left(\mathbf{x}_{0:k+1} - \mathbf{x}_{0:k+1}^l\right) p\left(\mathbf{z}_{k+1}|\mathbf{x}_{k+1}^l\right). \quad \text{(A10)}$$

Note that the choice of transition density as the importance function is only used as a tool to derive the particle-filter version of the conditional PCRLB. For the purpose of state estimation, any appropriate importance density function can be chosen for the particle filter.

APPENDIX B
APPROXIMATION OF $B_k^{22,b}$ BY PARTICLE FILTERS IN SECTION IV-B

If the measurement noise is additive Gaussian noise, the likelihood function can be written as follows:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \frac{1}{(2\pi)^{\frac{n_z}{2}}|R_k|^{\frac{1}{2}}} \times \exp\left\{-\frac{1}{2}\left[\mathbf{z}_k - h_k(\mathbf{x}_k)\right]^T R_k^{-1}\left[\mathbf{z}_k - h_k(\mathbf{x}_k)\right]\right\} \quad \text{(B1)}$$

where n_z is the dimension of the measurement vector $\mathbf{z}_k$. Taking the logarithm of the likelihood function, we have

$$-\log p(\mathbf{z}_k|\mathbf{x}_k) = c_0 + \frac{1}{2}\left(\mathbf{z}_k - h_k(\mathbf{x}_k)\right)^T R_k^{-1}\left(\mathbf{z}_k - h_k(\mathbf{x}_k)\right)$$

where c_0 denotes a constant independent of $\mathbf{x}_k$ and $\mathbf{z}_k$. Then the first and second-order partial derivatives of $\log p(\mathbf{z}_k|\mathbf{x}_k)$ can be derived respectively as follows:

$$\nabla_{\mathbf{x}_k} \log p(\mathbf{z}_k|\mathbf{x}_k) = \left[\nabla_{\mathbf{x}_k} h_k(\mathbf{x}_k)\right] R_k^{-1}\left(\mathbf{z}_k - h_k(\mathbf{x}_k)\right) \quad \text{(B2)}$$

$$-\Delta_{\mathbf{x}_k}^{\mathbf{x}_k} \log p(\mathbf{z}_k|\mathbf{x}_k) = \left[\nabla_{\mathbf{x}_k} h_k(\mathbf{x}_k)\right] R_k^{-1}\left[\nabla_{\mathbf{x}_k}^T h_k(\mathbf{x}_k)\right] - \Delta_{\mathbf{x}_k}^{\mathbf{x}_k} h_k(\mathbf{x}_k)\widetilde{\Sigma}_{\mathbf{v}_k}^{-1}\Upsilon_k^{22,b} \quad \text{(B3)}$$

where

$$\widetilde{\Sigma}_{\mathbf{v}_k}^{-1} = \begin{bmatrix} R_k^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & R_k^{-1} & \mathbf{0} & \vdots \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & R_k^{-1} \end{bmatrix}_{(n_x n_z)\times(n_x n_z)} \quad \text{(B4)}$$

and

$$\Upsilon_k^{22,b} = \begin{bmatrix} \mathbf{z}_k - h_k(\mathbf{x}_k) & \dots & \mathbf{0} \\ \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{z}_k - h_k(\mathbf{x}_k) \end{bmatrix}_{(n_x n_z)\times n_x}. \quad \text{(B5)}$$

Now with (A10) and (B3), we have

$$\begin{aligned} B_k^{22,b} &= E_{p_{k+1}^c}\left\{-\Delta_{\mathbf{x}_{k+1}}^{\mathbf{x}_{k+1}} \log p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})\right\} \\ &\approx \frac{1}{N}\sum_{l=1}^{N} \nabla_{\mathbf{x}_{k+1}} h(\mathbf{x}_{k+1}) R_{k+1}^{-1} \\ &\quad \times \nabla_{\mathbf{x}_{k+1}}^T h(\mathbf{x}_{k+1})|_{\mathbf{x}_{k+1}=\mathbf{x}_{k+1}^l} \end{aligned} \quad \text{(B6)}$$

where the identity

$$E_{p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})}\left\{\Upsilon_{k+1}^{22,b}\right\} = \mathbf{0} \quad \text{(B7)}$$

has been used.

ACKNOWLEDGMENT

The authors would like to thank Dr. K. G. Mehrotra, Dr. C. K. Mohan, and Dr. O. Ozdemir for helpful discussions and suggestions and the anonymous reviewers for their constructive comments.

REFERENCES

[1] P. Tichavsky, C. H. Muravchik, and A. Nehorai, "Posterior Cramér–Rao bounds for discrete-time nonlinear filtering," *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1386–1396, May 1998.

[2] H. L. Van Trees, *Detection, Estimation and Modulation Theory*. New York: Wiley, 1968, vol. 1.

[3] M. L. Hernandez, T. Kirubarajan, and Y. Bar-Shalom, "Multisensor resource deployment using posterior Cramér–Rao bounds," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 40, no. 2, pp. 399–416, Apr. 2004.

[4] K. Punithakumar, T. Kirubarajan, and M. L. Hernandez, "Multisensor deployment using PCRLBS, incorporating sensor deployment and motion uncertainties," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, no. 4, pp. 1474–1485, Oct. 2006.

[5] R. Tharmarasa, T. Kirubarajan, and M. L. Hernandez, "Large-scale optimal sensor array management for multitarget tracking," *IEEE Trans. Syst., Man, Cybern. C, Appl. Rev.*, vol. 37, no. 5, pp. 803–814, Sep. 2007.

[6] R. Tharmarasa, T. Kirubarajan, M. L. Hernandez, and A. Sinha, "PCRLB-based multisensor array management for multitarget tracking," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 2, pp. 539–555, Apr. 2007.

[7] M. Hurtado, T. Zhao, and A. Nehorai, "Adaptive polarized waveform design for target tracking based on sequential Bayesian inference," *IEEE Trans. Signal Process.*, vol. 56, no. 3, pp. 1120–1133, Mar. 2008.

[8] S. P. Sira, A. Papandreou-Suppappola, and D. Morrell, "Dynamic configuration of time-varying waveforms for agile sensing and tracking in clutter," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3207–3217, Jul. 2007.

[9] L. Zuo, "Conditional PCRLB for target tracking in sensor networks," Ph.D. dissertation, Syracuse Univ., Dept. Electr. Eng. Comput. Sci., , Syracuse, NY, Dec. 2010.

[10] L. Zuo, R. Niu, and P. K. Varshney, "Posterior CRLB based sensor selection for target tracking in sensor networks," in *Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Honolulu, HI, Apr. 2007, vol. 2, pp. 1041–1044.

[11] L. Zuo, R. Niu, and P. K. Varshney, "A sensor selection approach for target tracking in sensor networks with quantized measurements," in *Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Las Vegas, NV, Mar. 2008, pp. 2521–2524.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York: Cambridge Univ. Press, 1985.

[13] N. J. Gordon, D. J. Salmond, and A. F. M. Smith, "Novel approach to nonlinear/non-Gaussian Bayesian state estimation," *Proc. Inst. Electr. Eng. F—Radar Signal Process.*, vol. 140, no. 2, pp. 107–113, Apr. 1993.

[14] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Trans. Signal Process.*, vol. 50, no. 2, pp. 174–188, Feb. 2002.

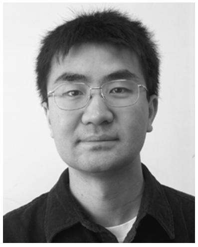
[15] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation With Applications to Tracking and Navigation*. New York: Wiley, 2001.

[16] G. Kitagawa, "Non-Gaussian state-space modeling of nonstationary time series," *J. Amer. Statist. Assoc.*, vol. 82, no. 400, pp. 1032–1063, Dec. 1987.

[17] J. H. Kotecha and P. M. Djuric, "Gaussian particle filtering," *IEEE Trans. Signal Process.*, vol. 51, no. 10, pp. 2592–2601, Oct. 2003.

Long Zuo received the B.S. degree from Xi'an Jiaotong University, Xi'an, China, in 1999 and the M.S. degree from the Institute of Automation, Chinese Academy of Sciences, Beijing, China, in 2002, both in electrical engineering.

Since 2002, he has been working towards the Ph.D. degree in electrical engineering at Syracuse University, Syracuse, NY. His research interests are in the areas of statistical signal processing, data fusion, and their applications to sensor networks.



Ruixin Niu (M'04) received the B.S. degree from Xi'an Jiaotong University, Xi'an, China, in 1994, the M.S. degree from the Institute of Electronics, Chinese Academy of Sciences, Beijing, China, in 1997, and the Ph.D. degree from the University of Connecticut, Storrs, in 2001, all in electrical engineering.

He is currently a Research Assistant Professor with Syracuse University, Syracuse, NY. His research interests are in the areas of statistical signal processing and its applications, including detection, estimation, data fusion, sensor networks, communications, and image processing.

Dr. Niu received the Best Paper Award, at the Seventh International Conference on Information Fusion in 2004. He is a coauthor of the paper that won the Best Student Paper Award at the Thirteenth International Conference on Information Fusion in 2010. He is the Associate Administrative Editor of the *Journal of Advances in Information Fusion*, and an Associate Editor of the *International Journal of Distributed Sensor Networks*.

Pramod K. Varshney (F'97) was born in Allahabad, India, on July 1, 1952. He received the B.S. degree in electrical engineering and computer science (with highest honors) and the M.S. and Ph.D. degrees in electrical engineering from the University of Illinois at Urbana-Champaign in 1972, 1974, and 1976, respectively.

Since 1976, he has been with Syracuse University, Syracuse, NY, where he is currently a Distinguished Professor of electrical engineering and computer science. His current research interests are in distributed sensor networks and data fusion, detection and estimation theory, wireless communications, image processing, radar signal processing, and remote sensing. He has published extensively. He is the author of *Distributed Detection and Data Fusion* (New York: Springer-Verlag, 1997).

Dr. Varshney was a James Scholar, a Bronze Tablet Senior, and a Fellow while with the University of Illinois. He is a member of Tau Beta Pi and is the recipient of the 1981 ASEE Dow Outstanding Young Faculty Award. He was the Guest Editor of the Special Issue on Data Fusion of the *Proceedings of the IEEE*, January 1997. In 2000, he received the Third Millenium Medal from the IEEE and the Chancellor's Citation for Exceptional Academic Achievement at Syracuse University. He serves as a Distinguished Lecturer for the IEEE Aerospace and Electronic Systems (AES) Society. He is on the editorial boards of the *International Journal of Distributed Sensor Networks* and the IEEE TRANSACTIONS ON SIGNAL PROCESSING. He was the President of the International Society of Information Fusion during 2001.