Total Perfect Codes in Tensor Products of Graphs

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Abstract. A total perfect code in a graph is a subset of the graph's vertices with the property that each vertex in the graph is adjacent to exactly one vertex in the subset. We prove that the tensor product of any number of simple graphs has a total perfect code if and only if each factor has a total perfect code.

1 Introduction

A *total perfect code* in a simple graph $G = (V(G), E(G))$ is a subset $C \subseteq V(G)$ with the property that for each $x \in V(G)$, the neighborhood $N(x) = \{y \mid xy \in E(G)\}$ contains exactly one vertex in C. If x is adjacent to $y \in C$, we say x is *covered* by y. This is illustrated with the graph in Figure 1, where the dark vertices form a total perfect code. Each vertex is covered by exactly one member of the code. Observe that many graphs (complete graphs K_n with $n \geq 3$, for instance) do not admit total perfect codes.

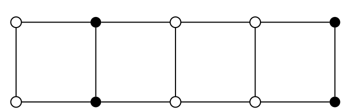

Figure 1

Total perfect codes have been studied in [1], [2], [3], [4] and [6] and appear in the literature under various names: efficient open domination sets, total domination sets and exact transversals. There is a complete characterization of total perfect codes for grid graphs in [6] and [2]. In this note we characterize total perfect codes for tensor products of graphs in terms of total perfect codes of their factors.

The *tensor product* of graphs G and H is the graph $G \otimes H$ whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are $(g,h)(g',h')$ where $gg' \in E(G)$ and $hh' \in E(H)$. As an example of a tensor product, Figure 2 shows $P_4 \otimes P_3$, where P_n denotes the path on n vertices. The tensor product is also known in the literature as the direct, Kronecker or categorical product and is often denoted by $\times$ rather than $\otimes$. A full treatment of this product can be found in [5]. If $G_1, G_2, \ldots, G_n$ are graphs, the *n-fold tensor product* $\bigotimes_{i=1}^n G_i = G_1 \otimes G_2 \otimes \cdots \otimes G_n$ consists of vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, where $(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)$ is an edge exactly when $x_i y_i \in E(G_i)$ for each $1 \leq i \leq n$. This is equivalent to the inductive definition $\bigotimes_{i=1}^n G_i = G_1 \otimes (\bigotimes_{i=2}^n G_i)$. The graphs G_i are called *factors* of the product. We denote by π_i the projection $\pi_i : V(\bigotimes_{i=1}^n G_i) \to V(G_i)$, defined by $\pi_i(x_1, x_2, \ldots, x_n) = x_i$.

2 Results

In this section we examine the relationship between total perfect codes in n-fold tensor products and total perfect codes of their factors. We give a constructive proof that an n-fold tensor product has a total perfect code if and only if all of its factors have total perfect codes. Our first proposition proves one direction.

Proposition 2.1 *Suppose $G_1, G_2, \ldots, G_n$ are graphs and G_i has total perfect code $C_i \subseteq V(G_i)$ for $1 \leq i \leq n$. Then $C_1 \times C_2 \times \cdots \times C_n$ is a total perfect code for $G_1 \otimes G_2 \otimes \cdots \otimes G_n$.*

Proof. Suppose that $C_i \subseteq V(G_i)$ is a total perfect code for G_i for $1 \leq i \leq n$. Form the Cartesian product $C = C_1 \times C_2 \times \cdots \times C_n$. We claim that C is a total perfect code for $G_1 \otimes G_2 \otimes \cdots \otimes G_n$.

Let $(g_1, g_2, \ldots, g_n) \in V(G_1 \otimes G_2 \otimes \cdots \otimes G_n)$. Then each g_i is adjacent to some $g_i' \in C_i$. Thus, $(g_1, g_2, \ldots, g_n)$ is adjacent to $(g_1', g_2', \ldots, g_n') \in C$, so each vertex of $G_1 \otimes G_2 \otimes \cdots \otimes G_n$ is covered by some element of C.

Now suppose there exists some $(g_1, g_2, \ldots, g_n) \in V(G_1 \otimes G_2 \otimes \cdots \otimes G_n)$ that is covered by two distinct elements in C, say $(g_1', g_2', \ldots, g_n')$ and $(g_1'', g_2'', \ldots, g_n'')$. This implies that $g_i g_i', g_i g_i'' \in E(G_i)$ with both g_i' and g_i'' in C_i for $1 \leq i \leq n$. Choose an index i for which $g_i' \neq g_i''$, and we see that vertex $g_i \in G_i$ is covered by distinct elements g_i' and g_i'' in the total perfect code C_i, a contradiction. Hence each vertex of $G_1 \otimes G_2 \otimes \cdots \otimes G_n$ is covered by exactly one element of C, so C is a total perfect code for $G_1 \otimes G_2 \otimes \cdots \otimes G_n$. ■

Figure 2 is an illustration of Proposition 2.1. Total perfect codes C_1 and C_2 (dark vertices) are indicated on paths P_4 and P_3 to the bottom and left of the product $P_4 \otimes P_3$. Observe that $C_1 \times C_2$ is a total perfect code for $P_4 \otimes P_3$.

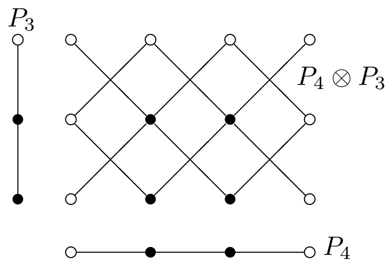

Figure 2

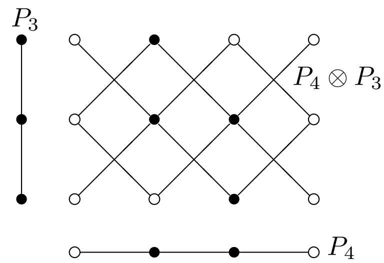

Figure 3

We will now prove a converse to Proposition 2.1: If a tensor product has a total perfect code, then each factor has a total perfect code. Ideally, we would hope that a reverse process of the proof of Proposition 2.1 would work, that is, given a total perfect code in the product, project it to a total perfect code in each factor. However, Figure 3 reveals the situation to be more intricate. A total perfect code for $P_4 \otimes P_3$ is indicated, but it does not project to a total perfect code on P_3. Clearly, some care is required here. The following proposition shows that projections of appropriate subsets of a total perfect code produce total perfect codes in the factors.

Proposition 2.2 *Suppose $G = G_1 \otimes G_2 \otimes \cdots \otimes G_n$ has a total perfect code C. If $(g_1, g_2, \ldots, g_n) \in V(G)$, then for any $1 \le i \le n$, the set $C_i = \pi_i\left(C \cap [N(g_1) \times N(g_2) \times \cdots \times N(g_{i-1}) \times V(G_i) \times N(g_{i+1}) \times \cdots \times N(g_n)]\right)$ is a total perfect code in G_i.*

Proof. Let $X = N(g_1) \times \cdots \times N(g_{i-1}) \times V(G_i) \times N(g_{i+1}) \times \cdots \times N(g_n)$. We want to show that $C_i = \pi_i(C \cap X)$ is a total perfect code in G_i. Take an arbitrary vertex x of G_i and observe that x is covered by some element of C_i as follows. The vertex $(g_1, g_2, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_n)$ of G must be covered by some element $(g'_1, g'_2, \ldots, g'_i, \ldots, g'_n) \in C$. Necessarily, $g'_k \in N(g_k)$ for $1 \le k \le n$ and $k \ne i$, so $(g'_1, g'_2, \ldots, g'_i, \ldots, g'_n) \in C \cap X$. Thus $g'_i \in \pi_i(C \cap X) = C_i$ and x is covered by $g'_i \in C_i$.

Now, if x were also covered by some $g''_i \in C_i$, there would be a vertex $(g''_1, g''_2, \ldots, g''_i, \ldots, g''_n) \in C \cap X$ that covers $(g_1, g_2, \ldots, x, \ldots, g_n)$. Then $(g_1, g_2, \ldots, x, \ldots, g_n)$ would be covered by both $(g'_1, g'_2, \ldots, g'_i, \ldots, g'_n)$ and $(g''_1, g''_2, \ldots, g''_i, \ldots, g''_n)$ in C. Hence, $g'_k = g''_k$ for all $1 \le k \le n$. In particular, $g'_i = g''_i$, so x is covered by exactly one element of C_i. Thus C_i is a total perfect code in G_i. ■

Propositions 2.1 and 2.2 imply that a tensor product of graphs has a total perfect code if and only if each factor has a total perfect code. In fact, we have the following stronger result.

Theorem 2.1 *Let $G_1, G_2, \ldots, G_n$ be graphs and let $G = G_1 \otimes G_2 \otimes \cdots \otimes G_n$. Then*

1. *G has exactly one total perfect code if and only if each factor G_i for $1 \le i \le n$ has exactly one total perfect code.*
2. *G has more than one total perfect code if and only if each factor G_i for $1 \le i \le n$ has at least one total perfect code and one factor has more than one total perfect code.*

Proof. Observe that Part 1 of the theorem follows from Part 2 together with Propositions 2.1 and 2.2, thus it suffices to prove only Part 2.

Suppose that G has two total perfect codes C and D. By Proposition 2.2, each G_i has at least one total perfect code. We now show that G_k for some $1 \le k \le n$ has two total perfect codes. Choose $(g_1, g_2, \ldots, g_n) \in V(G)$ that is adjacent to $(g_1', g_2', \ldots, g_n') \in C$ and $(g_1'', g_2'', \ldots, g_n'') \in D$ with $g_k' \ne g_k''$ for some k. Then by Proposition 2.2, the sets $C_k = \pi_k(C \cap [N(g_1) \times N(g_2) \times \cdots \times N(g_{k-1}) \times V(G_k) \times N(g_{k+1}) \times \cdots \times N(g_n)])$ and $D_k = \pi_k(D \cap [N(g_1) \times N(g_2) \times \cdots \times N(g_{k-1}) \times V(G_k) \times N(g_{k+1}) \times \cdots \times N(g_n)])$ are total perfect codes for G_k. Note that $g_k' \in C_k$ and $g_k'' \in D_k$ by construction, but $g_k'' \notin C_k$ for otherwise the vertex g_k in G_k is covered by both g_k' and g_k'' in C_k. Thus $C_k \ne D_k$ and G_k has at least two total perfect codes.

Conversely, suppose that each factor G_i for $1 \le i \le n$ has a total perfect code C_i and some factor has more than one total perfect code. Without loss of generality, assume that G_1 has two total perfect codes C_1 and C_1'. It follows from Proposition 2.1 that $C_1 \times C_2 \times \cdots \times C_n$ and $C_1' \times C_2 \times \cdots \times C_n$ are distinct total perfect codes for G. ■

We mention one application of these results. Klostermeyer and Goldwasser [6] chacterize the values of m and n for which the Cartesian product $P_m \times P_n$ of two paths admits a total perfect code. Even with just two factors, the situation is remarkably rich. By contrast, Propositions 2.1 and 2.2 make the analogous problem for the tensor product relatively simple, and we can state a result not just for the product of m paths, but for cycles as well. It was shown in [4] that a path P_n has a total perfect code if and only if $n \not\equiv 1 \pmod 4$, and in [3] that an n-cycle Z_n has a total perfect code if and only if $n \equiv 0 \pmod 4$. Thus Theorem 2.1 implies the following corollary.

Corollary 2.1 *A product $(\bigotimes_{i=1}^{m} P_{p_i}) \otimes (\bigotimes_{i=1}^{n} Z_{q_i})$ has a total perfect code if and only if $p_i \not\equiv 1 \pmod 4$ for $1 \le i \le m$ and $q_i \equiv 0 \pmod 4$ for $1 \le i \le n$.*

Despite Theorem 2.1, it is not possible in general to determine the number of total perfect codes in a product from the number of total perfect codes in its factors. This is illustrated in figures 4(a) and 4(b). For clarity, only one component of each product is shown; in each case the missing component is isomorphic to the one drawn.

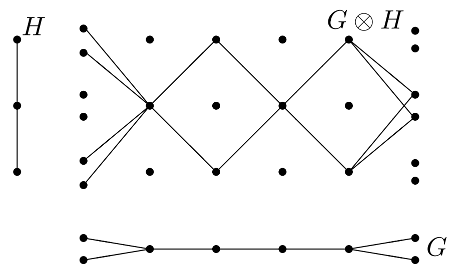

Figure 4(a)

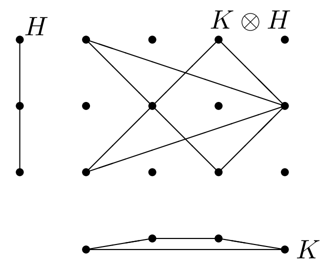

Figure 4(b)

In each case, the factor H admits exactly two total perfect codes. Factors G and K each admit four total perfect codes, as follows. Any code in G consists of two adjacent vertices incident with one of the two edges on the far left, together with two adjacent vertices incident with one of the two edges on the far right, for a total of four distinct codes. Any code in K consists of just two vertices incident with any of the four edges. But observe that $G \otimes H$ admits more codes than does $K \otimes H$. Any code in the indicated component of $G \otimes H$ consists of a choice of two vertices incident with any one of the four edges on the far left, together with two vertices incident with any one of the four edges on the far right, for a total of 16 codes. The other component of $G \otimes H$ also has 16 codes, so all together $G \otimes H$ admits 256 distinct codes. But any code in the indicated component of $K \otimes H$ consists of just two vertices incident with any one of the eight edges. Likewise the other component of $K \otimes H$ has eight distinct codes, so all together $K \otimes H$ has only 64 codes.

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