Perfect $r$-Codes in Strong Products of Graphs

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Abstract. A perfect $r$-code in a graph is a subset of the graph’s vertices with the property that each vertex in the graph is within distance $r$ of exactly one vertex in the subset. We prove that the $n$-fold strong product of simple graphs has a perfect $r$-code if and only if each factor has a perfect $r$-code.

1 Introduction

For a positive integer $r$, a perfect $r$-code in a simple graph $G = (V(G), E(G))$ is a subset $C$ of $V(G)$ for which the balls of radius $r$ centered at the vertices of $C$ form a partition of $V(G)$. This idea, introduced in [1], generalizes the notion of a standard error-correcting code. Perfect $r$-codes have also been used to model the problem of efficient placement of resources in a network. If the vertices in the code represent locations of resources, then every vertex in the graph is within distance $r$ of exactly one resource. Aside from applications, the study of perfect $r$-codes is an interesting combinatorial problem unto itself.

Perfect codes appear naturally in products of graphs. Perfect Hamming codes can be understood as perfect $r$-codes in Cartesian products of complete graphs and perfect Lee codes as perfect $r$-codes in Cartesian products.
of cycles. The most elusive problem in this area concerns perfect codes in the Lee metric. In [4] Golomb and Welch conjectured the nonexistence of \(n\)-dimensional perfect codes in the Lee metric for \(n \geq 3\) and \(r \geq 2\). This conjecture was only partially confirmed, see [11, 5, 6, 12] for details. Other results concerning perfect \(r\)-codes in Cartesian products appear in [2, 3].

Recently a number of authors have studied perfect \(r\)-codes in direct products [8, 9, 10, 13]. Perfect \(r\)-codes in a third type of product called the strong product have received little or no attention. Our note is a response to this deficiency. We prove constructively that an \(n\)-fold strong product has a perfect \(r\)-code if and only if each of its factors has a perfect \(r\)-code. This result is used to characterize which products of cycles and paths have perfect \(r\)-codes.

The distance between vertices \(u\) and \(v\) in \(G\), denoted by \(d_G(u, v)\), is the number of edges in a shortest path from \(u\) to \(v\). For a vertex \(v \in V(G)\), let \(B(v, r) = \{u \in V(G) | d_G(u, v) \leq r\}\) denote the \(r\)-ball centered at \(v\). Thus, a subset \(C \subseteq V(G)\) is a perfect \(r\)-code in \(G\) if \(\{B(c, r) | c \in C\}\) forms a partition of \(V(G)\). If \(x \in B(c, r)\), where \(c \in C\), we say \(x\) is \(r\)-dominated by \(c\). In other words, a perfect \(r\)-code in a simple graph \(G\) is a subset \(C\) of \(V(G)\) such that every vertex of \(G\) is \(r\)-dominated by exactly one vertex in \(C\). For example, in Figure 1, the dark vertices form a perfect 3-code. Each vertex is 3-dominated by exactly one member of the code. The 3-balls centered at the dark vertices are indicated by dotted lines. It is a simple matter to show that any two perfect \(r\)-codes in a given graph have the same cardinality.

The strong product of graphs \(G\) and \(H\) is the graph \(G \boxtimes H\) whose vertex set is the Cartesian product \(V(G) \times V(H)\) and whose edges are the pairs \((g, h)(g', h')\) of distinct vertices for which one of the following holds:

1. \(g = g'\) and \(hh' \in E(H)\)
2. \(gg' \in E(G)\) and \(h = h'\)
3. \(gg' \in E(G)\) and \(hh' \in E(H)\).

The graphs \(G\) and \(H\) are called factors of the product. The strong product
also appears in literature as the **strong direct product** or symmetric composition. As an example of a strong product, Figure 2 shows $P_3 \oslash C_4$ where $P_n$ denotes the path on $n$ vertices and $C_n$ is a cycle on $n$ vertices.

![Figure 2](image_url)

For clarity, the edges of form (1) and (2) are displayed bold. We note that edges of these types form what is called the **Cartesian** product of $G$ and $H$. The edges of form (3) yield the **direct** or **tensor** product of $G$ and $H$. (See [7] for details.) Thus the edges of $G \otimes H$ are the union of the edges of the Cartesian and direct products.

The strong product is associative in the sense that the map $((g_1, (g_2, g_3)) \mapsto ((g_1, g_2), g_3)$ is an isomorphism from $G_1 \oslash (G_2 \oslash G_3)$ to $(G_1 \oslash G_2) \oslash G_3$. Thus $G_1 \oslash G_2 \oslash \cdots \oslash G_2$ is well-defined without regard to grouping of factors, so it is natural to drop the parentheses. Doing this leads to the following fact which we accept as the definition of an $n$-fold strong product.

The $n$-fold strong product $\oslash_{i=1}^n G_i = G_1 \oslash G_2 \oslash \cdots \oslash G_n$ consists of vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ where a pair $(g_1, g_2, \ldots, g_n)(g_1', g_2', \ldots, g_n')$ of distinct vertices is an edge exactly when $g_i = g_i'$ or $g_i g_i' \in E(G_i)$ for each $1 \leq i \leq n$.

By [7, Lemma 2.1] the distance between two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the graph $G = \oslash_{i=1}^n G_i$ is

$$d_G(u, v) = \max_{1 \leq i \leq n} d_{G_i}(u_i, v_i).$$

In what follows $\pi_i$ denotes the usual projection functions, $\pi_i : V(\oslash_{i=1}^n G_i) \to V(G_i)$ defined by $\pi_i(g_1, g_2, \ldots, g_n) = g_i$. For more details on the strong product see [7].

## 2 Results

In this section we examine the relationship between perfect $r$-codes in the $n$-fold strong product of graphs and perfect $r$-codes in their factors. We show
that an \( n \)-fold strong product of graphs has a perfect \( r \)-code if and only if each factor has a perfect \( r \)-code. We start by proving the converse.

**Proposition 2.1** Suppose \( G_1, G_2, \ldots, G_n \) are graphs and \( G_i \) has a perfect \( r \)-code \( C_i \subseteq V(G_i) \) for \( 1 \leq i \leq n \). Then \( C_1 \times C_2 \times \cdots \times C_n \) is a perfect \( r \)-code in \( G_1 \bowtie G_2 \bowtie \cdots \bowtie G_n \).

**Proof.** Set \( G = G_1 \bowtie G_2 \bowtie \cdots \bowtie G_n \). Suppose that \( C_i \subseteq V(G_i) \) is a perfect \( r \)-code in \( G_i \) for \( 1 \leq i \leq n \). Form the Cartesian product \( C = C_1 \times C_2 \times \cdots \times C_n \).

We claim that \( C \) is a perfect \( r \)-code in \( G \).

Let \( g = (g_1, g_2, \ldots, g_n) \in V(G) \). Then each \( g_i \in V(G_i) \) is within a distance \( r \) of some \( c_i \in C_i \). Let \( c = (c_1, c_2, \ldots, c_n) \). Then by (1), \( d_{G_i}(g_i, c_i) \leq r \). Hence every vertex in \( G \) is \( r \)-dominated by a vertex in \( C_1 \times C_2 \times \cdots \times C_n \).

Now suppose there exists some \( g = (g_1, g_2, \ldots, g_n) \in V(G) \) that is \( r \)-dominated by vertices \( c = (c_1, c_2, \ldots, c_n) \) and \( c' = (c_1', c_2', \ldots, c_n') \) in \( C \). Then \( d_{G_i}(g_i, c_i) \leq r \) and \( d_{G_i}(g_i, c_i') \leq r \) for code elements \( c_i, c_i' \in C_i \). Hence \( c_i = c_i' \), so \( c = c' \). Thus \( C \) is a perfect \( r \)-code in \( G \).

Figure 3a illustrates Proposition 2.1 where the dark vertices belong to perfect \( 2 \)-codes in the factors and product. Indeed, the Cartesian product of the codes in the factors is a code in the product. However, as illustrated in Figure 3b, not every code in a product is a Cartesian product of codes in the factors. Thus a simple projection of a code in the product to a factor will not always produce a code in the factor. The next proposition shows how to construct codes in the factors from a code in the product.

**Proposition 2.2** Let \( G_1, G_2, \ldots, G_n \) be graphs and let \( G = G_1 \bowtie G_2 \bowtie \cdots \bowtie G_n \). Suppose that \( C \) is a perfect \( r \)-code in \( G \) and fix \( (g_1, \ldots, g_n) \in V(G) \).
For $1 \leq i \leq n$, set $D_i = \{(x_1, \ldots, x_n) \in V(G) | d_{G_i}(g_j, x_j) \leq r$ for $j \neq i\}$. Then $C_i = \pi_i(C \cap D_i)$ is a perfect $r$-code in $G_i$ for each $1 \leq i \leq n$.

**Proof.** Suppose $C$ is a perfect $r$-code in $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$ and let $v$ be any vertex in $G_i$. Since $G$ has perfect $r$-code $C$, the vertex $(g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n)$ in $G$ must be $r$-dominated by some $(c_1, c_2, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n) \in C$. Thus

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c_1, c_2, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n)) \leq r,$$

which by (1) implies that

$$\max\{d_{G_1}(g_1, c_1), d_{G_2}(g_2, c_2), \ldots, d_{G_i}(v, c_i), \ldots, d_{G_n}(g_n, c_n)\} \leq r. \quad (2)$$

Thus, $d_{G_i}(g_j, c_j) \leq r$ for each $j \neq i$ so $(c_1, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n) \in C \cap D_i$. Hence $c_i \in \pi_i(C \cap D_i) = C_i$. But (2) also implies $d_{G_i}(v, c_i) \leq r$, hence $v$ is $r$-dominated by $c_i \in C_i$.

Now suppose that $v$ is $r$-dominated by two elements $c_i$ and $c'_i$ in $C_i$. Then $c_i = \pi_i((c_1, c_2, \ldots, c_n))$ where $(c_1, c_2, \ldots, c_n) \in C \cap D_i$ and $c'_i = \pi_i((c'_1, c'_2, \ldots, c'_n))$ where $(c'_1, c'_2, \ldots, c'_n) \in C \cap D_i$. Thus, by definition of $D_i$ we have, $d_{G_i}(g_j, c_j) \leq r$ and $d_{G_i}(g_j, c'_j) \leq r$ for $j \neq i$. By assumption, $d_{G_i}(v, c_i) \leq r$ and $d_{G_i}(v, c'_i) \leq r$. Therefore by (1),

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c_1, c_2, \ldots, c_n)) \leq r$$

and

$$d_G((g_1, g_2, \ldots, g_{i-1}, v, g_{i+1}, \ldots, g_n), (c'_1, c'_2, \ldots, c'_n)) \leq r.$$

This implies that $(c_1, \ldots, c_n) = (c'_1, \ldots, c'_n)$, in particular $c_i = c'_i$. Thus $C_i$ is a perfect $r$-code in $G_i$ for every $1 \leq i \leq n$. ■

**Theorem 2.1** Suppose $G_1, G_2, \ldots, G_n$ are graphs and let $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. Then $G$ has more than one perfect $r$-code if and only if each factor $G_i$, for $1 \leq i \leq n$ has at least one perfect $r$-code and one factor has more than one perfect $r$-code.

**Proof.** Suppose that $G$ has perfect $r$-codes $C$ and $C'$. Then by Proposition 2.2, $G_i$ has at least one perfect $r$-code for each $1 \leq i \leq n$. We show now that $G_k$, for some $1 \leq k \leq n$, has two perfect $r$-codes. Since $G$ has two perfect $r$-codes there must exist a vertex $(g_1, \ldots, g_n)$ in $G$ that is $r$-dominated by $(c_1, \ldots, c_n) \in C$ and $(c'_1, \ldots, c'_n) \in C'$ where $c_k \neq c'_k$ for some $1 \leq k \leq n$. By Proposition 2.2, $C_k = \pi_k(C \cap D_k)$ and $C'_k = \pi_k(C \cap D'_k)$ are perfect $r$-codes in $G_k$. Notice that $C_k$ and $C'_k$ are not equal. Clearly $c_k \in C_k$ and $c'_k \in C'_k$, however, $c_k \notin C'_k$, for this would imply that $g_k \in G_k$ would be $r$-dominated by $c_k$ and $c'_k$ in $C'_k$. Thus, $C_k$ and $C'_k$ are distinct $r$-codes in $G_k$. 

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Conversely, suppose that $G_i$ has perfect $r$-code $C_i$ for each $1 \leq i \leq n$. Suppose also that $C'_1$ is another perfect $r$-code in $G_1$. Then by Proposition 2.1, $C_1 \times C_2 \times \cdots \times C_n$ and $C'_1 \times C_2 \times \cdots \times C_n$ are both perfect $r$-codes in $G$.

Although we have the above theorem, Figures 4a and 4b illustrate that it is not possible to determine the number of perfect $r$-codes in a strong product based on the number of perfect $r$-codes in the factors. The graph $G$ admits two perfect 2-codes and the graphs $H$ and $K$ both admit three perfect 2-codes. However, $H \boxtimes G$ has six perfect 2-codes formed by the Cartesian product of the codes in the factors while $K \boxtimes G$ has twelve perfect 2-codes, six coming from the Cartesian products of the codes in the factors and six more that contain vertices in a staggered pattern. The dark vertices in $K \boxtimes G$ indicate one of the staggered perfect 2-codes.

In a series of papers [8, 9, 10, 13], Jerebic, Jha, Klavžar, Špacapan and Žerovnik characterize the conditions under which a direct product of cycles admits a perfect $r$-code. The situation is remarkably complex. By contrast, our propositions show the analogous problem for the strong product is quite simple. In fact, we can state a result not just for the strong product of cycles, but paths as well. It is simple to check that, for a given $r$, the cycle $Z_s$ on $s$ vertices admits a perfect $r$-code if and only if $s$ is a multiple of $2r + 1$, and that any path $P_t$ admits a perfect $r$-code no matter the value of $t$.

**Corollary 2.1** A product $(\boxtimes_{i=1}^m Z_{s_i}) \boxtimes (\boxtimes_{i=1}^n P_{t_i})$ admits a perfect $r$-code if and only if each $s_i$ is a multiple of $2r + 1$.

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References


