A QUASI CANCELLATION PROPERTY FOR THE DIRECT PRODUCT OF GRAPHS

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Abstract. We are motivated by the following question concerning the direct product of graphs. If $A \times C \cong B \times C$, what can be said about the relationship between A and B? If cancellation fails, what properties must A and B share? We define a structural equivalence relation $\sim$ (called similarity) on graphs, weaker than isomorphism, for which $A \times C \cong B \times C$ implies $A \sim B$. Thus cancellation holds, up to similarity. Moreover, if C is bipartite, then $A \times C \cong B \times C$ if and only if $A \sim B$. We conjecture that the prime factorization of connected bipartite graphs is unique up to similarity of factors, and we offer some results supporting this conjecture.

1. Introduction

Let Γ_0 denote the class of graphs for which vertices are allowed to have loops. The *direct product* of two graphs A and B in Γ_0 is the graph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose edges are all pairs $(a, b)(a', b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. By interpreting aa', bb' and $(a, b)(a', b')$ as directed arcs from the left to the right vertex, the direct product can also be understood as a product on digraphs. In fact, since any graph can be identified with a symmetric digraph (where each edge is replaced by a double arc) the direct product of graphs is a special case of the direct product of digraphs. However, except where digraphs are needed in one proof, we restrict our attention to graphs.

The direct product enjoys a limited cancellation property. Lovász [8] proved that if C has an odd cycle, then $A \times C \cong B \times C$ if and only if $A \cong B$; further, cancellation also holds if C is arbitrary but there are homomorphisms $A \to C$ and $B \to C$. Since such homomorphisms certainly exist if A and B are bipartite (and C has at least one edge), then cancellation can only fail if C is bipartite and A and B are not both bipartite. (See [1, 5] for further results on cancellation for various graph products.)

Indeed, it is well known that the cancellation property does not hold in general. Figures 1(a) and 1(b) show an example. If P is the path on three vertices with loops at each end, then $K_3 \times K_2 \cong P \times K_2$, though $K_3 \ncong P$. (In fact, as our Proposition 2 will show, $K_3 \times C \cong P \times C$ for any bipartite graph C.)

This paper explores the following question. If $A \times C \cong B \times C$, what can be said about the relationship between A and B? If cancellation fails,

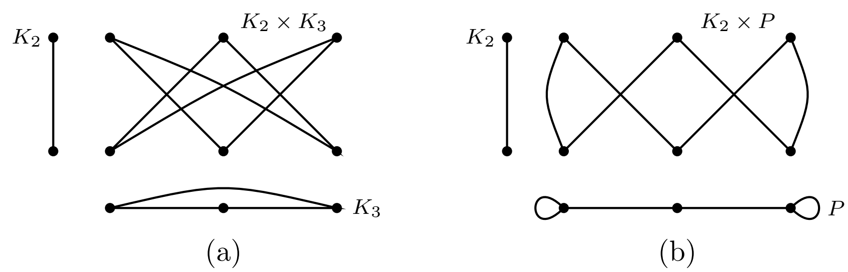

FIGURE 1. Failure of cancellation

what properties must A and B share? We define a structural relation $\sim$ on graphs, weaker than isomorphism, for which $A \times C \cong B \times C$ implies $A \sim B$. Thus cancellation holds, up to $\sim$, so this can be viewed as a kind of "quasi cancellation property." Further, we show that if C is bipartite, then $A \times C \cong B \times C$ if and only if $A \sim B$.

The reader is assumed to be familiar with the basic properties of direct products, including Weichsel's theorem on connectivity. See Chapter 5 of [6] for an excellent survey.

2. RESULTS

We begin with the following definition.

Definition 1. *Two graphs A and B in Γ_0 are said to be* similar, *written $A \sim B$, if there are bijections $\alpha, \beta : V(A) \to V(B)$ satisfying $aa' \in E(A)$ if and only if $\alpha(a)\beta(a') \in E(B)$.*

Remark 1. *By replacing α with α^{-1} we get the following equivalent formulation. Graphs A and B are similar if there are bijections $\alpha : V(B) \to V(A)$ and $\beta : V(A) \to V(B)$ satisfying $a\alpha(b) \in E(A)$ if and only if $\beta(a)b \in E(B)$. Despite the appealing symmetry of this alternate formulation, Definition 1 tends to be easier to use.*

Similarity is a weaker notion than isomorphism. Though $A \cong B$ imples $A \sim B$, the converse is not true in general. Figures 2(a) and 2(b) give examples of similar graphs. The thin solid and dashed lines represent the effects of α and β respectively. Figure 2(a) shows $K_3 \sim P$, where P is the path on three vertices with loops on each end. Figure 2(b) shows $C_6 \sim 2K_3$.

It is straightforward to check that similarity is an equivalence relation. Before investigating its further properties, we need a lemma linking cancellation of $A \times C \cong B \times C$ to cancellation of $A \times K_2 \cong B \times K_2$.

Lemma 1. *Suppose A, B and C are graphs and C has at least one edge. Then $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$.*

Proof. Given digraphs X and Y, let $\hom(X, Y)$ be the number of homomorphisms from X to Y. We will use the following theorem ([4], Theorem

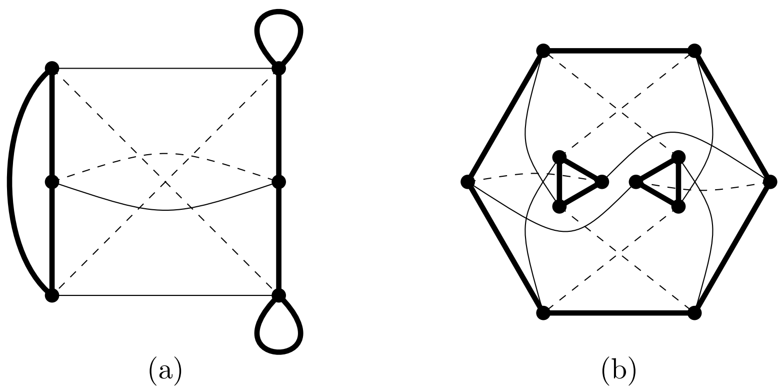

FIGURE 2. Pairs of similar graphs

2.11) of Lovász: If D and D' are digraphs, then $D \cong D'$ if and only if $\hom(X, D) = \hom(X, D')$ for all digraphs X. We will also use the fact ([4], Corollary 2.3) that $\hom(X, C \times D) = \hom(X, C) \hom(X, D)$.

Identify A, B, C and K_2 with their symmetric digraphs (i.e. each edge is replaced with a double arc). If we can show $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$ for the symmetric digraphs, then certainly this holds for the underlying graphs as well.

Suppose $A \times C \cong B \times C$. Multiplying both sides by K_2 and using associativity and commutativity of $\times$, we get $(A \times K_2) \times C \cong (B \times K_2) \times C$. Let X be a digraph. Then

$$\begin{aligned} \hom(X, A \times K_2) \hom(X, C) &= \hom(X, (A \times K_2) \times C) \\ &= \hom(X, (B \times K_2) \times C) \\ &= \hom(X, B \times K_2) \hom(X, C) \end{aligned}$$

If X is bipartite (i.e. if its underlying graph is bipartite) then $\hom(X, C) \neq 0$ because the map sending two partite sets to the two endpoints of a double arc of C is a homomorphism. Thus $\hom(X, A \times K_2) = \hom(X, B \times K_2)$. On the other hand, if X is not bipartite, then there can be no homomorphism from X to a bipartite graph, and hence $\hom(X, A \times K_2) = 0 = \hom(X, B \times K_2)$. Thus $\hom(X, A \times K_2) = \hom(X, B \times K_2)$ for any X, so Lovász's theorem gives $A \times K_2 \cong B \times K_2$. □

We interpret Lemma 1 to mean that cancellation never fails badly. Although $A \times C \cong B \times C$ does not imply $A \cong B$ in general, at worst the factor of C can be replaced with the small graph K_2. And $A \times K_2 \cong B \times K_2$ is not far from $A \cong B$. In fact, $A \times K_2 \cong B \times K_2$ is equivalent to $A \sim B$, as the next two propositions show.

Proposition 1. *Suppose A, B and C are arbitrary graphs, and C has at least one edge. Then $A \times C \cong B \times C$ implies $A \sim B$.*

Proof. Suppose $A \times C \cong B \times C$. Then by Lemma 1, $A \times K_2 \cong B \times K_2$. Set $V(K_2) = \{0, 1\}$. Fix an isomorphism $\theta : A \times K_2 \to B \times K_2$, and let its component-wise expression be $\theta(a, \varepsilon) = (\theta_1(a, \varepsilon), \theta_2(a, \varepsilon))$ for appropriate homomorphisms $\theta_1 : A \times K_2 \to B$ and $\theta_2 : A \times K_2 \to K_2$.

The proof is quite simple except for one technical detail, namely that θ can be chosen so $\theta(V(A) \times \{0\}) = V(B) \times \{0\}$ and $\theta(V(A) \times \{1\}) = V(B) \times \{1\}$. For the moment, assume that θ has this property. Observe that it implies that $\theta_2(a, 0)\theta_2(a', 1)$ is the edge of K_2 for every $a, a' \in V(A)$.

Define maps $\alpha, \beta : V(A) \to V(B)$ as $\alpha(a) = \theta_1(a, 0)$ and $\beta(a) = \theta_1(a, 1)$. It is easy to confirm that these are bijections. Notice that

$$\begin{aligned} aa' \in E(A) &\iff (a, 0)(a', 1) \in E(A \times K_2) \\ &\iff \theta(a, 0)\theta(a', 1) \in E(B \times K_2) \\ &\iff (\theta_1(a, 0), \theta_2(a, 0))(\theta_1(a', 1), \theta_2(a', 1)) \in E(B \times K_2) \\ &\iff \theta_1(a, 0)\theta_1(a', 1) \in E(B) \\ &\iff \alpha(a)\beta(a') \in E(B). \end{aligned}$$

Therefore $A \sim B$.

To complete the proof, we must show that θ can be chosen so that $\theta(V(A) \times \{\varepsilon\}) = V(B) \times \{\varepsilon\}$ for each $\varepsilon \in \{0, 1\}$. Let $\overline{\varepsilon} = 1 - \varepsilon$, that is $\overline{0} = 1$ and $\overline{1} = 0$.

A few preliminary remarks are in order. Observe that if G is a connected non-bipartite graph, then $G \times K_2$ is a connected bipartite graph with partite sets $V(G) \times \{0\}$ and $V(G) \times \{1\}$, and the map $\mu_G : G \times K_2 \to G \times K_2$, defined as $\mu(g, \varepsilon) = (g, \overline{\varepsilon})$, is an automorphism that interchanges the partite sets. By contrast, if H is a connected bipartite graph, then $H \times K_2$ has exactly two components, each isomorphic to H.

Now suppose A has a bipartite component H, and B has a bipartite component H', and there is an isomorphism $\gamma : H \to H'$. Then $H \times K_2$ is a pair of components of $A \times K_2$, and $H' \times K_2$ is a pair of components of $B \times K_2$, and there is an isomorphism $H \times K_2 \to H' \times K_2$ defined as $(h, \varepsilon) \mapsto (\gamma(h), \varepsilon)$. We are free to assume θ restricts to this isomorphism on $H \times K_2$.

Next, let $H_1, H_2, \ldots, H_n$ be a maximal set (possibly empty) of bipartite components of A for which B has bipartite components $H'_1, H'_2, \ldots, H'_n$ with $H_i \cong H'_i$ for $1 \leq i \leq n$. As in the above paragraph, we may assume θ restricts to isomorphisms $H_i \times K_2 \to H'_i \times K_2$ with $\theta(V(H_i) \times \{\varepsilon\}) = V(H'_i) \times \{\varepsilon\}$.

Now let X be a component of $A \times K_2$ that is not one of the components of the $H_i \times K_2$. Then either X is a component of $H \times K_2$ where H is a bipartite component of A that is not isomorphic to any component of B, or $X = G \times K_2$ for some non-bipartite component G of A. Consider these cases separately.

First suppose X is a component of $H \times K_2$ where H is a bipartite component of A that is not isomorphic to any component of B. Then $\theta(X)$ cannot be a component of $H' \times K_2$ where H' is a bipartite component of B, because then $H \cong X \cong \theta(X) \cong H'$, contradicting the fact that B has no subgraph isomorphic to H. Thus $\theta(X) = G' \times K_2$ where G' is a non-bipartite component of B. Now one partite set of X consists of vertices of form $(x, 0)$ and the other consists of vertices of form $(y, 1)$. Since θ preserves the partite sets, it sends one partite set of X to $V(G') \times \{0\}$ and the other to $V(G') \times \{1\}$. Since there is an automorphism $\mu_{G'} : G' \times K_2 \to G' \times K_2$ that interchanges the partite sets of $G' \times K_2$, we may assume $\theta(x, \varepsilon) \in V(B) \times \{\varepsilon\}$ for every vertex (x, ε) of X.

Finally suppose $X = G \times K_2$ where G is a non-bipartite component of A. Since θ preserves partite sets, we must have either $\theta(V(G) \times \{\varepsilon\}) \subseteq V(B) \times \{\varepsilon\}$ or $\theta(V(G) \times \{\varepsilon\}) \subseteq V(B) \times \{\overline{\varepsilon}\}$. In the event of the latter case, since there is an automorphism μ_G that interchanges the partite sets of X, we can assume $\theta(V(G) \times \{\varepsilon\}) \subseteq V(B) \times \{\varepsilon\}$.

This completes the demonstration that we can choose θ with $\theta(V(A)\times\{\varepsilon\}) = V(B) \times \{\varepsilon\}$ for $\varepsilon \in \{0, 1\}$, so the proof is complete. □

Proposition 2. *Suppose C is an arbitrary bipartite graph that has at least one edge. Then $A \times C \cong B \times C$ if and only if $A \sim B$.*

Proof. The necessity follows from Proposition 1. Conversely suppose $A \sim B$, so there are bijections $\alpha, \beta : V(A) \to V(B)$ satisfying $aa' \in E(A)$ if and only if $\alpha(a)\beta(a') \in E(B)$. Let C_0 and C_1 be partite sets of C, and define $\theta : A \times C \to B \times C$ as follows.

$$\theta(a, c) = \begin{cases} (\alpha(a), c) & \text{if } c \in C_0 \\ (\beta(a), c) & \text{if } c \in C_1 \end{cases}$$

Observe that θ is an isomorphism: Take an arbitrary edge $(a, c)(a', c')$ in $E(A \times C)$. We may assume $c \in C_0$ and $c' \in C_1$. Then

$$\begin{aligned} (a, c)(a', c') \in E(A \times C) &\iff aa' \in E(A) \text{ and } cc' \in E(C) \\ &\iff \alpha(a)\beta(a') \in E(B) \text{ and } cc' \in E(C) \\ &\iff (\alpha(a), c)(\beta(a'), c') \in E(B \times C) \\ &\Longrightarrow \theta(a, c)\theta(a', c') \in E(B \times C). \end{aligned}$$

Conversely, suppose $\theta(a, c)\theta(a', c') \in E(B \times C)$. From the definition of θ, it follows that $cc' \in E(C)$. By interchanging the order of the endpoints of the edge $\theta(a, c)\theta(a', c')$ and relabeling (if necessary) we may thus assume $c \in C_0$ and $c' \in C_1$. Then the above implications can be reversed to obtain $(a, c)(a', c') \in E(A \times C)$. Therefore θ is an isomorphism from $A \times C$ to $B \times C$. □

Figure 3 illustrates Proposition 2. Figure 3(a) indicates that $A \sim B$, where A is the 4-cycle and B is two copies of an edge with loops at each end. Let C be the (bipartite) path on three vertices. According to figures 3(b) and 3(c), products $A \times C$ and $B \times C$ are isomorphic, as each consists of two copies of $K(2, 4)$. (For clarity, one component is drawn bold.) For another example, figures 1(a) and 1(b) show $K_3 \times K_2 \cong P \times K_2$, while Figure 2(a) shows $K_3 \sim P$.

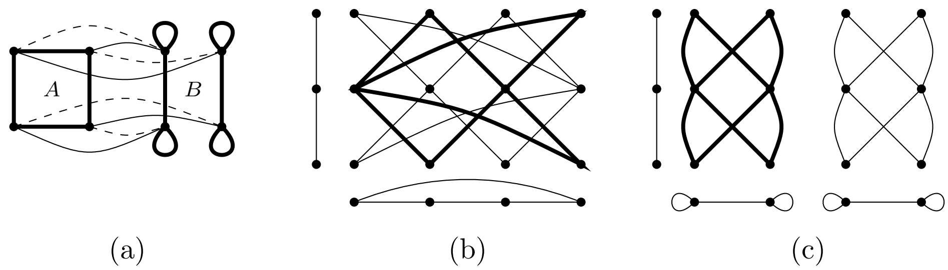

FIGURE 3. An illustration of Proposition 2

Proposition 3. *If $A \sim A'$ and $B \sim B'$, then $A \times B \sim A' \times B'$.*

Proof. Since $B \sim B'$, Proposition 2 gives $B \times K_2 \cong B' \times K_2$. Observe that $A \times (B \times K_2) \cong A' \times (B' \times K_2)$ for the following reason: If $B \times K_2$ has no edges, then $A \times (B \times K_2)$ and $A' \times (B' \times K_2)$ are totally disconnected graphs with the same number of vertices, so they are isomorphic. If $B \times K_2$ has at least one edge, then $A \times (B \times K_2) \cong A' \times (B' \times K_2)$ by Proposition 2. Then $(A \times B) \times K_2 \cong (A' \times B') \times K_2$, and Proposition 2 applied again gives $A \times B \sim A' \times B'$. □

Proposition 4. *Suppose C is an arbitrary graph with at least one edge. Then $A \sim B$ if and only if $A \times C \sim B \times C$.*

Proof. Using Proposition 2 twice, $A \sim B$ if and only if $A \times (C \times K_2) \cong B \times (C \times K_2)$, if and only if $(A \times C) \times K_2 \cong (B \times C) \times K_2$, if and only if $A \times C \sim B \times C$. □

For an arbitrary graph A, let $[A] = \{G \in \Gamma_0 : G \sim A\}$ be the similarity equivalence class containing A. Proposition 3 means that the direct product is a well-defined operation on equivalence classes as $[A] \times [B] = [A \times B]$. But more can be said. Proposition 4 means that there is a general cancellation law for the equivalence classes, namely that if $[C]$ is a class that contains graphs with edges, then $[A] \times [C] = [B] \times [C]$ if and only if $[A] = [B]$. (For $[A] = [B]$ if and only if $A \sim B$, if and only if $A \times C \sim B \times C$, if and only if $[A \times C] = [B \times C]$, if and only if $[A] \times [C] = [B] \times [C]$.)

Thus, we can view similarity as correcting a defect in the properties of the direct product. Though cancellation fails in general, we can take a courser point of view and look at equivalence classes rather than graphs, and cancellation holds. It is natural to ask what other properties may be gained with this point of view. We conclude with a discussion of bipartite graphs and a conjecture about their prime factorizations.

3. BIPARTITE GRAPHS

This section explores some consequences of applying Proposition 2 to bipartite graphs. In what follows, let $+$ indicate disjoint union of graphs, and let $2A = A + A$. Our first result shows that in the class of bipartite graphs, similarity is the same as isomorphism.

Proposition 5. *Suppose A and B are bipartite and $A \sim B$. Then $A \cong B$.*

Proof. By Proposition 2 we have $A \times K_2 \cong B \times K_2$. But A is bipartite, so $A \times K_2 \cong 2A$. Similarly $B \times K_2 \cong 2B$. Thus $2A \cong 2B$, so $A \cong B$. □

Thus any similarity equivalence class contains at most one bipartite graph. However, it is of course possible that a bipartite graph is similar to a non-bipartite graph, as Figure 2(b) shows. Our next proposition explores this possibility.

Proposition 6. *Let B be a connected bipartite graph. The following are equivalent.*

(a) *B is similar to a non-bipartite graph.*
(b) *B is similar to a disconnected graph.*
(c) *$B \sim (A_1 + A_2)$ where A_1 and A_2 are connected non-bipartite graphs and $A_1 \sim A_2$.*
(d) *$B \sim 2A$ for some connected non-bipartite graph A.*
(e) *$B \cong A \times K_2$ for some connected non-bipartite graph A.*
(f) *B admits an involution that interchanges its partite sets. (An involution is an atomorphism that is its own inverse.)*

Proof. Assume Statement (a), so $B \sim A$ for a non-bipartite graph A. Proposition 2 gives $B \times K_2 \cong A \times K_2$. Now, $B \times K_2$ has two components, so $A \times K_2$ has two components. If A were connected, then $A \times K_2$ would have only one component, so A is disconnected. This is Statement (b).

Now assume Statement (b), so $B \sim A$, where A is disconnected. Then $A \times K_2$ has at least two components. In fact, $A \times K_2 \cong B \times K_2 \cong 2B$, so $A \times K_2$ has exactly two components. If A had more than two components, then $A \times K_2$ would have more than two components, so A has exactly two components, which we call A_1 and A_2. Then $B \sim A = A_1 + A_2$. Each of A_1 and A_2 is non-bipartite, for otherwise $A \times K_2$ would have more than two components. We just need to confirm $A_1 \sim A_2$. Observe $2B \cong A \times K_2 \cong (A_1 + A_2) \times K_2 \cong (A_1 \times K_2) + (A_2 \times K_2)$. From $2B \cong (A_1 \times K_2) + (A_2 \times K_2)$ it follows that $B \cong A_1 \times K_2 \cong A_2 \times K_2$. Then $A_1 \sim A_2$ by Proposition 2. We have Statement (c).

Assume Statement (c). Since $A_1 \sim A_2$, it follows that $A_1 + A_2 \sim A_1 + A_1 = 2A_1$. Then $B \sim 2A_1$, which is Statement (d).

Assume Statement (d), that is $B \sim 2A$ for some connected non-bipartite A. Then $2B \cong B \times K_2 \cong 2A \times K_2 \cong 2(A \times K_2)$. From $2B \cong 2(A \times K_2)$ it follows that $B \cong A \times K_2$, which is Statement (e).

Assume Statement (e), that is $B \cong A \times K_2$. Then $(a, \varepsilon) \mapsto (a, \overline{\varepsilon})$ is an involution of $A \times K_2$ that interchanges its partite sets.

Finally, assume B admits an involution β that interchanges its partite sets. Define a graph A with $V(A) = V(B)$ and $E(A) = \{b\beta(b') : bb' \in$

$E(B)\}$. Then every edge of A has both endpoints in the same partite set of B, so A is disconnected. Observe $B \sim A$: Let α the identity map on the set $V(A) = V(B)$. Then $bb' \in E(B)$ implies $b\beta(b') \in E(A)$, so $\alpha(b)\beta(b') \in E(A)$. Conversely, if $\alpha(b)\beta(b') = b\beta(b') \in E(A)$, then the definition of A means either $bb' \in E(B)$ or $\beta^{-1}(b)\beta(b') \in E(B)$. In the latter case, $\beta^{-1}(b)\beta(b') = \beta(b)\beta(b')$, so $bb' \in E(B)$. Either way, $bb' \in E(B)$, so $B \sim A$. Using a now-familiar trick, this gives $B \times K_2 \cong A \times K_2$, so $A \times K_2$ has exactly two components, and hence the disconnected graph A must be non-bipartite. This brings us full circle to Statement (a). □

Proposition 5 and parts (a) and (f) of Proposition 6 reveal that it is relatively rare for a bipartite graph to be similar to a different graph. For a connected bipartite graph B, the class $[B]$ contains only B unless B possesses a very special kind of symmetry, namely that it admits an involution that interchanges its partite sets.

This sort of symmetry plays a key role in the article [7] by Jha, Klavžar and Zmazek. They show that for connected bipartite graphs A and B, the product $A \times B$ has isomorphic components if one of A or B admits an automorphism that interchanges its partite sets. They conjecture the converse to be true. The converse was proved in [2] for the special case that A and B have no 4-cycles, and the preprint [3] has a general proof of the converse. We interpret $A \times B$ having isomorphic components as equivalent to the equation $A \times B \cong K_2 \times C$ for some graph C, and this equation can hold even if just one of A and B (say B) is bipartite. In this case, it is natural to ask what structure the equation $A \times B \cong K_2 \times C$ forces on the bipartite graph B. The methods of [3] appear to apply to this case, and they suggest that B has an involution that interchanges its partite sets. In light of Proposition 6(e,f) this means B has K_2 as a factor.

In other words, if K_2 divides a connected bipartite graph that factors as $A \times B$ (with B bipartite), then K_2 divides B. Said differently, if K_2 appears in one prime factorization of a connected bipartite graph then it appears in every prime factorization. Now, it is well known that the class of connected non-bipartite graphs in Γ_0 obeys unique prime factorization. But this does not hold in the class of connected bipartite graphs, so it is interesting that a factor of K_2 appears to be unique. This causes us to wonder if prime factorization of connected bipartite graphs is unique up to similarity. Indeed, that is precisely the case in the example illustrated in Figure 1. We conjecture that this is true in general.

Conjecture 1. *Suppose a connected bipartite graph has prime factorizations $B \times P_1 \times P_2 \times \cdots \times P_m$ and $B' \times P'_1 \times P'_2 \times \cdots \times P'_n$ where B and B' are bipartite and all other factors are non-bipartite. Then $m = n$, $B \cong B'$, and the remaining factors can be reindexed so $P_i \sim P'_i$.*

As an illustration of this conjecture, Figure 4 shows two prime factorizations of the logo for the Sixth Slovenian International Conference on Graph Theory. The reader may check that the two non-bipartite factors are similar.

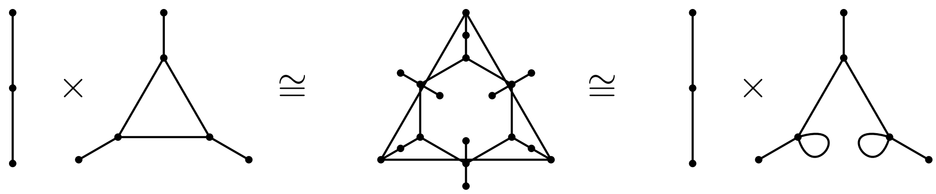

FIGURE 4. Two prime factorizations of a graph

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