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Graph Bases and Diagram Commutativity

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Abstract Given two cycles A and B in a graph, such that $A \cap B$ is a non-trivial path, the *connected sum* $A \hat{+} B$ is the cycle whose edges are the symmetric difference of $E(A)$ and $E(B)$. A special kind of cycle basis for a graph, a *connected sum basis*, is defined. Such a basis has the property that a hierarchical method, building successive cycles through connected sum, eventually reaches all the cycles of the graph. It is proved that every graph has a connected sum basis. A property is said to be *cooperative* if it holds for the connected sum of two cycles when it holds for the summands. Cooperative properties that hold for the cycles of a connected sum basis will hold for all cycles in the graph. As an application, commutativity of a groupoid diagram follows from commutativity of a connected sum basis for the underlying graph of the diagram. An example is given of a noncommutative diagram with a (non-connected sum) basis of cycles which do commute.

Keywords Cycle basis · Connected sum · Commutative diagram · Groupoid · Robust cycle basis · Ear basis · Geodesic cycle

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1 Introduction

We say a diagram of isomorphisms is **commutative** if the composition of the morphisms around each cycle in the underlying graph is the identity. (Here an arc traversed backwards corresponds to the inverse of the corresponding morphism.) It seems plausible that if such a diagram commutes on every cycle of a cycle basis for its underlying graph, then the diagram should actually be commutative. However, this is false, and we present a counterexample below, in Sect. 4. To understand how graph bases relate to commutativity, we introduce *connected sum bases*.

The connected sum of two topological manifolds is the manifold formed by deleting an open ball from the two summands and identifying the bounding spheres. This operation can be applied to graphs, as cycles are 1-manifolds. If two cycles in a graph intersect in a non-trivial path (1-ball), then their **connected sum** is the cycle that is the union of the two cycles minus the interior of the common path of their intersection. Therefore the operation of connected sum coincides with addition in the cycle space of the graph, but it is defined only when the two cycles being added have intersection equal to a non-trivial path.

A cycle basis for the cycle space of a graph G is called a *connected sum basis* (CS basis) if any cycle in G can be constructed by a sequence of connected sum operations using cycles in the basis, in a hierarchy to be described precisely in Sect. 2. In Sect. 3 we prove that every graph has a CS basis; Sect. 4 uses this to study commutativity of diagrams of invertible morphisms. Section 5 is a discussion.

We remark in passing that another reason to consider CS bases is that they give a solution to the 1-dimensional case of a more general topological problem: *Construct manifolds as connected sums of elementary components.* Oriented 2-manifolds, other than the sphere, are connected sums of tori. For 3-manifolds, a unique connected sum decomposition result was proved by Milnor [16]; Wall [20] gave a CS-decomposition for 1-connected 6-manifolds; cf. [13]. A graph-like 2-dimensional case is discussed in Sect. 5.

Another useful property of CS bases is that they guarantee *topological constancy* (all partial sums are cycles as a given cycle is built from the basis). The notion of topological constancy was introduced in [2] and independently in [7]. In [10] we noted two complexity invariants that might be investigated, maximum degree and number of connected components, and one may add cycle rank to this list. Connected sum bases allow one to construct cycles and even-degree subgraphs, so that the partial sums do not exceed the complexity of the target.

In addition to topological and graph theoretic aspects, CS bases also have algebraic implications. Topological constancy alone is not sufficient to control commutativity in a diagram. An example is given in [10] of two commutative squares that intersect in disjoint edges and have a non-commutative square as their cycle-space sum. In Sect. 4 we present a diagram which fails to commute in spite of the commutativity of all the cycles in a (non-CS) basis for the underlying graph. However, we prove that if a diagram commutes on a CS basis, then it commutes on all cycles.

Applications to related concepts in biochemistry were proposed by Klemm and Stadler [11]. In their approach, cycles represent microstates of some system and perturbing a cycle by the addition of a short cycle as a "detour" of low energy leads to a connection with Markov processes.

Previous work [7–12] used different terminology.

2 Connected Sum and Robustness of Cycle Bases

Let $G = (V, E)$ be a simple graph; $|G|$ denotes $|V|$, the order of G, and $\|G\|$ denotes its size $|E|$. A **cycle** is a 2-regular, connected graph. We write $\mathrm{Cyc}(G)$ for the set of all cycle subgraphs of G.

The cycle space of G, denoted $\mathbf{Z}(G)$, is the $\mathbb{F}_2$-vector space consisting of all edge-sets in G that induce even-degree subgraphs, with symmetric difference of sets as the sum operation. We identify edge-sets with the subgraphs they induce; e.g., $\mathrm{Cyc}(G) \subseteq \mathbf{Z}(G)$. The **cycle rank** of G is $m(G) := \dim(\mathbf{Z}(G))$. Then ([6, p. 39], [1, pp. 23–28])

$$m(G) = \|G\| - |G| + c_0(G), \tag{1}$$

where $c_0(G)$ is the number of connected components of G. A **cycle basis** of G is a basis of $\mathbf{Z}(G)$ consisting entirely of cycles. Any graph has a cycle basis because $\mathrm{Cyc}(G)$ spans $\mathbf{Z}(G)$ by Euler's result that every even-degree graph is an edge-disjoint union of cycles. See Harary [6] or Diestel [1] for undefined graph theory terminology.

Call two cycles **compatible** if they intersect in a nontrivial path, that is, a path with at least one edge. Given two cycles Z_1 and Z_2 of a graph G, the mod-2 sum $Z_1 + Z_2$ in $\mathbf{Z}(G)$ is a **connected sum (CS)** if and only if Z_1 and Z_2 are compatible. To indicate that the two cycles in a sum are compatible, we will write either

$$Z_1 \hat{+} Z_2 \quad \text{or} \quad Z_1 + Z_2 \ \ (\text{CS}).$$

The connected sum of two cycles is again a cycle. This is clear, but we derive it briefly with some useful notation. Let P be any nontrivial path contained in a cycle C. Let $C - P$ denote the **complementary path** Q obtained by deleting all edges and all interior vertices of P from C. Hence $C = P \cup Q$ and $P \cap Q = \overline{K_2}$, the graph with two isolated vertices. Now if $Z', Z'' \in \mathrm{Cyc}(G)$ and if $Z' \cap Z'' = P$, a nontrivial path, then $Z' + Z'' = P' \cup P''$, where $P' = Z' - P$ and $P'' = Z'' - P$ are the appropriate complementary paths. So $Z' \hat{+} Z'' \in \mathrm{Cyc}(G)$ because P' and P'' intersect only in their endpoints.

As a binary operation, $\hat{+}$ is commutative but not associative: both $Z_1 \hat{+} Z_2$ and $(Z_1 \hat{+} Z_2) \hat{+} Z_3$ could be defined, while $Z_2 \hat{+} Z_3$ is not defined if Z_3 intersects Z_1 but not Z_2. For $Z_i \in \mathscr{C} \subseteq \mathrm{Cyc}(G)$, we call a sequence $Z_1, Z_2, \ldots, Z_m$ a **connected sum sequence from** $\mathscr{C}$ if the partial sum $Z^{i-1} := Z_1 + \cdots + Z_{i-1}$ intersects the next term, Z_i, in a non-trivial path for all $2 \leq i \leq m$. By induction each partial sum Z^i is a cycle. We denote this state of affairs as

$$Z_1 + Z_2 + \cdots + Z_m \quad (\text{CS}). \tag{2}$$

Equivalently, one can write (2) as the left-most parenthesized sum:

$$\Big(\cdots\big((Z_1 \mathbin{\hat{+}} Z_2) \mathbin{\hat{+}} Z_3\big)\cdots\Big) \mathbin{\hat{+}} Z_m.$$

Given a set of cycles $\mathscr{C} \subseteq \mathrm{Cyc}(G)$, the **connected sum closure** of $\mathscr{C}$ is the set $\rho(\mathscr{C})$ of all cycles that are connected sums of sequences from $\mathscr{C}$. Define $\rho^0(\mathscr{C}) = \mathscr{C}$, $\rho^1(\mathscr{C}) = \rho(\mathscr{C})$, and $\rho^k(\mathscr{C}) = \rho(\rho^{k-1}(\mathscr{C}))$ for k any positive integer. Thus

$$\mathscr{C} \subseteq \rho^1(\mathscr{C}) \subseteq \rho^2(\mathscr{C}) \subseteq \rho^3(\mathscr{C}) \subseteq \cdots \subseteq \mathrm{Cyc}(G).$$

As G is finite, this chain eventually stabilizes (i.e., eventually the inclusions are all equality). It can stabilize at $\mathscr{C}$ itself if no two members of $\mathscr{C}$ are compatible, or at $\mathrm{Cyc}(G)$, or at some intermediate set of cycles. We say $\mathscr{C} \subseteq \mathrm{Cyc}(G)$ is a CS **generating set** if $\rho^k(\mathscr{C}) = \mathrm{Cyc}(G)$ for some k. The **depth** of $\mathscr{C}$ is the *least* such k.

Here is an example from [10]. Recall that a cycle is **geodesic** if there is no path between two of its vertices which is strictly shorter than their distance within the cycle; let $\mathscr{G}(G)$ denote the set of geodesic cycles in G.

Theorem 1 *If G is a graph, then $\mathscr{G}(G)$ is a* CS *generating set of depth no more than $c(G) - g(G)$, where $c(G)$ and $g(G)$ are the lengths of longest and shortest cycles in G.*

A **robust basis** for a graph G is a basis $\mathscr{B}$ for which $\rho(\mathscr{B}) = \mathrm{Cyc}(G)$. Robust bases have been constructed for several classes of graphs. The Mac Lane basis of a connected plane graph is robust [10]. Complete graphs [7] have robust bases, as do complete bipartite graphs $K_{r,s}$, when $\max(r, s) \leq 4$. However, [5] proves that $K_{n,n}$ has no robust basis when $n \geq 8$.

Call a basis a **connected sum basis** (CS basis) if it is a CS generating set, that is, $\rho^k(\mathscr{B}) = \mathrm{Cyc}(G)$ for some k. (We called such bases *iteratively robust* in [10].)

The next section proves that every graph has a CS basis.

3 Main Results

We show that every graph has a connected sum basis using *ear decompositions*. Given a graph G, consider a pairwise edge-disjoint sequence of subgraphs

$$C_1, P_2, P_3, \ldots, P_n, \tag{3}$$

where C_1 is a cycle and each P_i is a nontrivial path (called an *ear*). Let G_i be the union of the first i terms $(1 \leq i \leq n)$, that is,

$$G_i = C_1 \cup P_2 \cup P_3 \cup \ldots \cup P_i.$$

The sequence (3) is an **ear decomposition** of G if $G = G_n$, and for each ear, $G_{i-1} \cap P_i$ is the endpoints of P_i. A theorem of Whitney [22] asserts that a graph is 2-connected if and only if it has an ear decomposition.

Given an ear decomposition (3), we define an **associated ear basis** to be a cycle basis $\mathscr{B} = \{C_1, C_2, \ldots, C_n\}$ where $C_i - P_i$ is a path P_i' in G_{i-1} between the endpoints of P_i, for each $2 \leq i \leq n$. This is indeed a basis: It is linearly independent because each element contains an edge that belongs to no other element. Also, Eq. (1) implies $|\mathscr{B}| = m(G)$, because each P_i adds one more edge than vertex to G_{i-1}. An **ear basis** is a basis that is associated to an ear decomposition; see [4,14].

Theorem 2 *Let G be a 2-connected graph with cycle rank n and let $\mathscr{B}$ be an ear basis for G. Then $\mathscr{B}$ is a connected sum basis of depth at most $n-1$, that is,*

$$\rho^{n-1}(\mathscr{B}) = \mathrm{Cyc}(G). \tag{4}$$

Proof Suppose G is 2-connected. Take an ear decomposition $C_1, P_2, \ldots, P_n$, with a corresponding ear basis $\mathscr{B} = \{C_1, C_2, \ldots, C_n\}$. It is enough to establish (4), and for this it suffices to show $\mathrm{Cyc}(G) \subseteq \rho^{n-1}(\mathscr{B})$. We prove this by induction on n.

If $n = 1$, then $\mathrm{Cyc}(G) = \rho^0(\mathscr{B})$, as G is a cycle in this case. Let $n \geq 2$ and assume the theorem is true for graphs of cycle rank $n-1$. Note that $C_1, P_2, P_3, \ldots, P_{n-1}$ is an ear decomposition of G_{n-1} and $\mathscr{B}^* = \{C_1, C_2, \ldots, C_{n-1}\} \subseteq \mathscr{B}$ is a corresponding ear basis of G_{n-1}. By the induction hypothesis, any cycle in G_{n-1} belongs to $\rho^{n-2}(\mathscr{B}^*) \subseteq \rho^{n-2}(\mathscr{B}) \subseteq \rho^{n-1}(\mathscr{B})$.

We next show that any cycle Z in G that is not entirely contained in G_{n-1} must belong to $\rho^{n-1}(\mathscr{B})$. Such a Z necessarily contains P_n as a subpath, but the part of it in G_{n-1} may deviate from P_n'. (Recall $C_n = P_n' \cup P_n$, for a path P_n' in G_{n-1}.) See Fig. 1.

Denote the end vertices of P_n as a and b. Call a vertex x of Z **transverse** if x also belongs to C_n, and Z has a subpath wxy for which at least one of w or y does not belong to C_n. All transverse vertices of Z are in P_n' and hence in G_{n-1}. Vertices a and b may or may not be transverse, depending on Z. Moving along P_n' from a to b, list the transverse vertices consecutively as $x_1, x_2, \ldots, x_\ell$. (Possibly $x_1 = a$ or $x_\ell = b$. See Fig. 1.)

If $Z = C_n$, then $Z \in \mathscr{B}$, and $Z \in \rho^0(\mathscr{B}) \subseteq \rho^{n-1}(\mathscr{B})$. (In this case there are no transverse vertices.) Hence, one may suppose $Z \neq C_n$ and complete the proof by showing that Z can be written as a connected sum $Z = C_n + Z_1 + Z_2 + \cdots + Z_m$ (CS) for cycles Z_i in G_{n-1}. (For then (4) holds because C_n as well as each Z_i belong to $\rho^{n-2}(\mathscr{B})$.)

We define the cycles Z_i algorithmically. For Z_1, start at the first transverse vertex x_1, and move along a path Q_1 in $Z - P_n$ (counterclockwise in Fig. 1) until reaching a transverse vertex x_{i_1}. (Possibly $i_1 > 2$, as in Fig. 1.) Then move from x_{i_1} back along $P_n' = C_n - P_n$ to x_1, forming a path Q_1'. Put $Z_1 = Q_1 + Q_1'$.

Define Z_2 similarly: start at x_{i_1} and continue along $Z - P_n$ until reaching the first transverse vertex x_{i_2} with $i_2 > i_1$. Let Q_2 be the path traversed. From x_{i_2}, move back along $C_n - P_n$ to x_{i_1}, calling this path Q_2'. Put $Z_2 = Q_2 + Q_2'$.

We continue this pattern inductively. Say $Z_{p-1} = Q_{p-1} + Q_{p-1}'$ has been defined, where Q_{p-1} is a path along $Z - P_n$ from $x_{i_{p-2}}$ to $x_{i_{p-1}}$, with $i_{p-1} < \ell$, and Q_{p-1}' is a path along $C_n - P_n$ from $x_{i_{p-1}}$ to $x_{i_{p-2}}$. From $x_{i_{p-1}}$, move along $Z - P_n$ to the

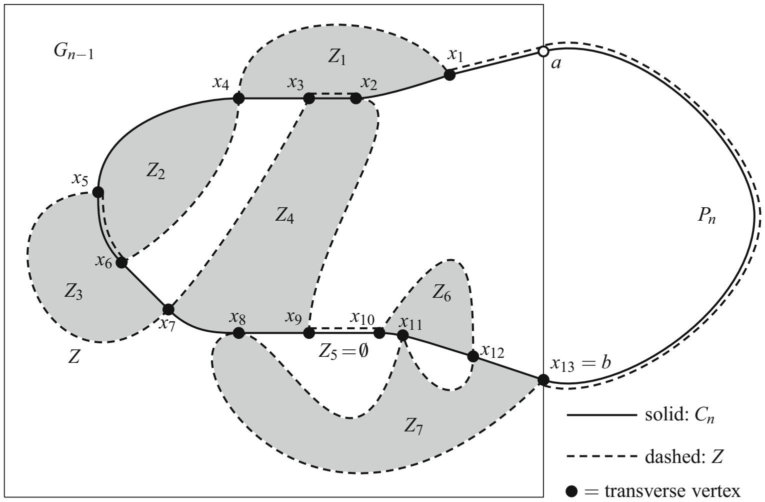


Fig. 1 Construction of the cycles Z_i. The cycle C_n is drawn solid, and Z is dashed

first transverse vertex x_{i_p} for which $i_p > i_{p-1}$. Denote this resulting path in Z as Q_p. Now move back along $C_n - P_n$ to $x_{i_{p-1}}$, forming a path Q'_p. Put $Z_p = Q_p + Q'_p$.

There is one degenerate situation, illustrated by Z_5 in Fig. 1. Here Q_5 happens never to deviate from C_n as it traverses Z from $x_{i_4} = x_9$ to the next transverse vertex $x_{i_5} = x_{10}$. Then Q'_5 is a path back from $x_{i_5} = x_{10}$ to $x_{i_4} = x_9$. In this case $Q_5 = Q'_5$. We agree that in the definition $Z_p = Q_p + Q'_p$, the sum represents addition in the *edge space* of G (i.e., symmetric difference on edge sets), so that $Z_p = \emptyset$ when $Q_p = Q'_p$.

At this point, one has constructed edge-disjoint paths $Q'_1, \ldots, Q'_m$ in C_n whose union is a path in G_{n-1} that runs along C_n from x_1 to x_ℓ. Let P be the path in C_n that runs from x_ℓ to x_1, completing the cycle, so that

$$C_n = P + Q'_1 + Q'_2 + \cdots + Q'_m.$$

Similarly, one has edge-disjoint paths $Q_1, \ldots, Q_m$ in Z whose union is a path in G_{n-1} that runs along Z from x_1 to x_ℓ. Then

$$Z = P + Q_1 + Q_2 + \cdots + Q_m.$$

Adding the two displayed equations above, we get

$$Z = C_n + Z_1 + Z_2 + \cdots + Z_m. \tag{5}$$

Assume without loss of generality that any term Z_i in this sum that is $\emptyset$ has been deleted and that the remaining terms are reindexed sequentially.

Next we claim that (5) is indeed a connected sum. By construction, $C_n \cap Z_1 = Q'_1$, which is a non-trivial path. Further, we claim that $(C_n + Z_1 + \cdots + Z_{i-1}) \cap Z_i = Q'_i$. Indeed, by construction Q'_i shares no edge with any $Q'_1, \ldots, Q'_{i-1}$, nor does it share an edge with any $Q_1, \ldots, Q_{i-1}$. Thus Q_i' is a subgraph of $C_n + Z_1 + Z_2 + \cdots + Z_{i-1}$. Similarly, Q_i shares no edge with any $Q_1, \ldots, Q_{i-1}$. It is possible that some internal vertex of Q_i intersects some Q_r' for which $r < i$. But any such intersection points are canceled by the term Z_r in the partial sum $C_n + Z_1 + \cdots + Z_{i-1}$. It follows that no internal vertex of Q_i belongs to $C_n + Z_1 + \cdots + Z_{i-1}$. From these considerations we infer that $(C_n + Z_1 + \cdots + Z_{i-1}) \cap Z_i = (C_n + Z_1 + \cdots + Z_{i-1}) \cap \left(Q_i + Q_i'\right) = Q_i'$.

We have now written an arbitrary cycle Z of G as

$$Z = C_n + Z_1 + Z_2 + \cdots + Z_m \quad (\text{CS}), \tag{6}$$

with each summand in $\rho^{n-2}(\mathscr{B})$. Therefore $Z \in \rho^{n-1}(\mathscr{B})$, so (4) and the theorem hold. □

If a graph is not 2-connected, then the union of cycle bases of its (2-connected) blocks is a cycle basis for the entire graph. (If any block is K_2, its cycle space is Ø, so also its cycle basis is empty.) Applying Theorem 2 to each block yields a corollary.

Corollary 1 *Every graph has a connected sum basis.*

In particular, the union of ear bases of all the blocks of a graph is a CS basis. A different improvement of Theorem 2 is possible which includes Theorem 1. The following construction was also proposed by Eppstein [3].

Theorem 3 *Every* 2*-connected graph has an ear basis of geodesic cycles.*

Proof Let C_1 be a geodesic cycle in a graph G. Inductively, having constructed $C_1, \ldots, C_{t-1}$, choose P_t to be an ear of minimum length for which the G_{t-1}-distance between the endpoints of P_t is minimum. Any purported short-cut for C_t produces a shorter choice than C_t, which is impossible. □

A subset $\mathscr{P} \subseteq \mathrm{Cyc}(G)$ is called a **cooperative property** provided that $Z_1, Z_2 \in \mathscr{P}$ implies $Z_1 \hat{+} Z_2 \in \mathscr{P}$ (for compatible Z_1 and Z_2). We say a set $\mathscr{D}$ of cycles in G **has property** $\mathscr{P}$ if $\mathscr{D} \subseteq \mathscr{P}$. Induction yields the following.

Theorem 4 *Let $\mathscr{P}$ be a cooperative property. If $\mathscr{C}$ is a* CS *generating set that has property $\mathscr{P}$, then $\mathscr{P} = \mathrm{Cyc}(G)$.*

Hence a CS basis possesses a cooperative property if and only if every cycle of the graph has this property. The next section shows commutativity in diagrams is cooperative.

4 Graphs, Diagrams, Commutativity, and Groupoids

Recall that a directed graph (or digraph) is a set of vertices and a set of ordered pairs of distinct vertices, called arcs. Given any graph, one may form a digraph by choosing for each edge exactly one of the two possible arcs; this digraph is called an **orientation of the graph**. One can also simply replace every edge by both the corresponding arcs;

Fig. 2 A *diagram* is a digraph in a category. Vertices are objects; arcs are morphisms

$$\begin{tikzcd}
 & A \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[rr] \arrow[dl] \arrow[dd] & & D \arrow[dl] \arrow[dd] \\
E \arrow[rr] \arrow[dd] & & F \arrow[rr] \arrow[dd] & & G \arrow[rr] \arrow[dd] & & H \arrow[dd] & \\
 & I \arrow[rr] \arrow[dl] & & J \arrow[rr] \arrow[dl] & & K \arrow[rr] \arrow[dl] & & L \arrow[dl] \\
M \arrow[rr] & & N \arrow[rr] & & O \arrow[rr] & & P &
\end{tikzcd}$$

we call this the **induced digraph**. Conversely, any digraph D has an underlying graph $U(D)$.

A **diagram** is a mapping of a directed graph into a category, $\delta : D \to \mathbf{C}$, where for each vertex v of D, $\delta(v)$ is an object of $\mathbf{C}$ and each arc f from v to w in D is represented by a $\mathbf{C}$-morphism between the corresponding vertex-objects:

$$\delta(f) : \delta(v) \to \delta(w);$$

equivalently, $\delta(f) \in \mathrm{Hom}_{\mathbf{C}}(\delta(v), \delta(w))$.

Each directed path (dipath) in D induces a morphism by composition. Call two distinct dipaths **parallel** if they have the same initial and terminal vertices. A diagram is said to **parallel commute** if parallel dipaths always induce the same morphism. That is, the message sent depends only on the initiating and terminating vertex and not on the route taken. See, e.g., Mac Lane [15, pp. 3–8].

However, as defined, parallel commutativity has a structural weakness. Consider the example of a digraph which is an orientation of the graph C_4 (the 4-cycle) so that arcs alternate in direction. Hence, all dipaths have length 1 and no parallel dipaths exist. Any diagram on such digraphs will parallel commute.

We are interested in the case where $\mathbf{C}$ is a **groupoid** category [15, p. 20], [19, pp. 45–50]; that is, *every morphism is invertible*. Let $\mathbf{G}$ be a fixed but arbitrary groupoid. In this case, one can strengthen the concept of commutativity. Extend every diagram $\delta : D \to \mathbf{G}$ to a diagram

$$\delta^* : D^* \to \mathbf{G}$$

where D^* is the induced digraph on the underlying graph G of D by setting

$$\delta^*(a^{-1}) = \delta(a)^{-1},$$

where a^{-1} on the left means the inverse arc, $(v, w)^{-1} = (w, v)$, while inverse on the right-hand side of the equality is in $\mathbf{G}$. We call δ **groupoid-commutative** if every directed cycle (dicycle) in D^* induces an identity morphism in $\mathbf{G}$. In this case, one says that **each cycle commutes** as the choice of orientation and of starting/ending point does not affect the commutativity of any dicycle representing the cycle.

Groupoid-commutativity implies parallel commutativity but not conversely. Hereafter we use "commutative" only to refer to groupoid-commutativity (Fig. 2).

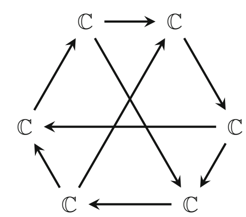


Fig. 3 A noncommutative diagram that commutes on a non-CS basis. Each morphism is rotation by $\frac{2\pi}{3}$

Theorem 5 *If G is any graph and $\delta : D \to \mathbf{G}$ is a diagram with $U(D) = G$ and if δ commutes on a* CS *basis for G, then δ is commutative on all cycles.*

By Theorem 4, it suffices to prove that commutativity is cooperative.

Lemma 1 *The connected sum of commutative cycles is commutative.*

Proof Let $\delta : D \to \mathbf{G}$ be a diagram and let H be the underlying graph of D. Suppose $Z_1, Z_2 \in \mathrm{Cyc}(H)$ with $Z_1 \cap Z_2 = P$, a non-trivial path. Let a, b denote the two endpoints of P. There are two dipaths corresponding to P, namely, P^+ from a to b and P^- from b to a. Let $P_1 := Z_1 - P$ and $P_2 := Z_2 - P$ with P_1^+ the orientation of P_1 from b to a and P_2^- the orientation of P_2 from a to b. If Z_1 and Z_2 commute, then letting "$\circ$" denote composition in $\mathbf{G}$ (written left-to-right) one has

$$\delta(Z_1^+) = \delta(P_1^+) \circ \delta(P^+) = 1_{\delta(a)} = \delta(P^-) \circ \delta(P_2^-) = \delta(Z_2^-),$$

where Z_1^+ and Z_2^- denote the corresponding orientations of Z_1, Z_2. Therefore,

$$1_{\delta(a)} = \delta(Z_1^+) \circ \delta(Z_2^-) = \delta(P_1^+) \circ \delta(P_2^-) = \delta(Z^+)$$

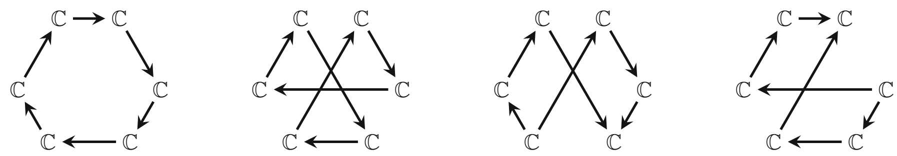
where $Z = Z_1 \hat{+} Z_2$ and Z^+ is the orientation of Z agreeing with P_1^+ and P_2^-. $\square$

To underscore the significance of Theorem 5, Example 1 below exhibits a noncommutative diagram in the groupoid of sets and bijections that commutes on a (non-CS) basis.

Example 1 An orientation of the complete bipartite graph $K_{3,3}$ is shown in Fig. 3. Consider the diagram δ that maps each vertex to $\mathbb{C}$, the set of complex numbers, and each properly traversed arc to clockwise rotation t of $2\pi/3$ around 0. The diagram is not commutative, as it does not commute on any of its nine C_4-subgraphs.

We demonstrate a cycle basis for $K_{3,3}$ on which the example diagram, Fig. 3, commutes. For this, let $\mathscr{B} = \{C_1, C_2, C_3, C_4\}$ be the four hexagons in the diagram that are shown in Fig. 4, where for clarity we have oriented the edges of the hexagons according to the diagram though of course these four cycles are unoriented graphs as elements of $\mathbf{Z}(K_{3,3})$. The cycles C_1 and C_2 commute because $t^6 = 1$, while C_3 and C_4 commute by cancellation.

To show $\mathscr{B}$ is a basis we just need to check independence as the underlying graph $K_{3,3}$ has cycle rank $9 - 6 + 1 = 4$. Let $\mathscr{C} = \{C_1, C_2\}$ and let $\mathscr{D} = \{C_3, C_4\}$. Certainly each of the sets $\mathscr{C}$ and $\mathscr{D}$ is independent. Note that any linear combination of the



C_1 C_2 C_3 C_4

Fig. 4 Four commutative cycles from Fig. 3. Their underlying subgraphs form a basis for $K_{3,3}$

elements of $\mathscr{C}$ has 0 or 3 diagonals, while any nontrivial linear combination of the elements of $\mathscr{D}$ has 2 diagonals. (A "diagonal" is any of the three edges not on the outer hexagon of Fig. 3.)

Now suppose $c_1C_1 + c_2C_2 + c_3C_3 + c_4C_4 = 0$. If $c_1 = c_2 = 0$, then $c_3 = c_4 = 0$ because $\mathscr{D}$ is independent. If $c_3 = c_4 = 0$, then $c_1 = c_2 = 0$ because $\mathscr{C}$ is independent. Otherwise $c_1C_1 + c_2C_2 = c_3C_3 + c_4C_4$, and each side of this equation has the same number of diagonals. By the previous paragraph this number of diagonals must be 0, and each c_i is zero. Thus independence holds.

Note that no two members of $\mathscr{B}$ are compatible, so $\rho(\mathscr{B}) = \mathscr{B}$. Commutativity can fail because $\mathscr{B}$ is not a CS-basis (so also not an ear basis).

5 Discussion

We have shown that groupoid diagrams commute if and only if every cycle in a CS basis commutes. This provides an efficient procedure to determine if a diagram of isomorphisms commutes. But if the basis is not a CS-basis, then commutativity for its cycles does not guarantee global commutativity.

Thus, logical checks on the consistency of complex systems may be wrong because of a hidden pathology in the combinatorics of the diagram and the basis. However, a reliable strategy, using connected sum bases, does guarantee commutativity of the entire diagram provided it holds on a basis.

Other properties are also cooperative. *Commutativity up to a natural equivalence* (CUTNE) was noted in [8]. Another cooperative property allows the groupoid hom-sets to have a natural involution, commutativity up to $\pm$-sign. Thus, when a groupoid diagram commutes or anti-commutes on all cycles in a CS basis, then every cycle commutes or anti-commutes.

In quantum-computing, a database modeled on the hypercube could have each of 2^d vertices v encoding a system-state, as an object $X(v)$ in a category. If any of the d bits of v is flipped, then there should be a morphism $X(v) \to X(w)$, where w is v with one of its bits reversed which gives the effect of changing a parameter. If one wishes to not lose information, all these morphisms should be isomorphisms. By testing only cycles in a basis, one obtains a very strong improvement over the task of checking *all* cycles. For example, a CS basis exists for the hypercube [10]; in the 5-dimensional case, one can test 49 cycles - rather than the total of 51 billion distinct cycle-subgraphs (A085408 in [17]).

One may represent a CS generating set $\mathcal{C}$ for a graph G using a spanning subgraph $\Gamma := \Gamma(G, \mathcal{C})$ of the Cayley graph [21, pp. 19–30] of the additive group of the $\mathbb{F}_2$-vector subspace $\mathbf{Z}(G)$. The vertex set of Γ is $\mathrm{Cyc}(G)$ while the edges at a vertex Z correspond to the cycles which are compatible with Z. More exactly, Γ is the nested union of an expanding sequence of subgraphs Γ_k corresponding to $\rho^k(\mathcal{C})$. The connectedness of Γ is equivalent to $\mathcal{C}$ being a generating set. A similar approach, though using only one subgraph, was proposed in [12,18]. The graph Γ models the evolution of cycles [9].

A different use of connected sum applies to 2-dimensional simplicial complexes. We call a 2-complex K **quasi-Eulerian** if *every edge lies in a positive even number of triangles* (which are the 2-simplexes). Let L be any 2-complex and define

$$\mathbf{Z}(L) := \{K \subseteq L : K \text{ is quasi-Eulerian}\}.$$

Note that $\mathbf{Z}(L)$ corresponds to the kernel of the boundary map from the 2-chains to the 1-chains and so is an $\mathbb{F}_2$-vector space (via symmetric difference of triangle-sets). A 2-dimensional **pseudo-manifold** is a 2-complex with every edge in *exactly two* triangles (plus uniformity and connectedness conditions, see [19, p. 148]). We ask:

Does every quasi-Eulerian K have a triangle-disjoint decomposition into pseudo-manifolds? Under what conditions on a 2-complex L does $\mathbf{Z}(L)$ have a basis of pseudo-manifolds? When is it possible to write any pseudo-manifold contained in L as a connected sum of those in a basis?

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