A CANCELLATION PROPERTY FOR THE DIRECT PRODUCT OF GRAPHS

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Abstract

Given graphs $A$, $B$ and $C$ for which $A \times C \cong B \times C$, it is not generally true that $A \cong B$. However, it is known that $A \times C \cong B \times C$ implies $A \cong B$ provided that $C$ is non-bipartite, or that there are homomorphisms from $A$ and $B$ to $C$. This note proves an additional cancellation property. We show that if $B$ and $C$ are bipartite, then $A \times C \cong B \times C$ implies $A \cong B$ if and only if no component of $B$ admits an involution that interchanges its partite sets.

Keywords: graph products, graph direct product, cancellation.

2000 Mathematics Subject Classification: 05C60.

1. Introduction

Denote by $\Gamma_0$ the class of graphs for which vertices are allowed to have loops. The direct product of two graphs $A$ and $B$ in $\Gamma_0$ is the graph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose edges are all pairs $(a,b)(a',b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. By interpreting $aa'$, $bb'$ and $(a,b)(a',b')$ as directed arcs from the left to the right vertex, the direct product can also be understood as a product on digraphs. In fact, since any graph can be identified with a symmetric digraph (where each edge is replaced by a double arc) the direct product of graphs is a special case of the direct product of digraphs. However, except where digraphs are needed in one proof, we restrict our attention to graphs.
The direct product obeys a limited cancellation property. Lovász [4] proved that if $C$ is not bipartite, then $A \times C \cong B \times C$ if and only if $A \cong B$. He also proved cancellation holds if $C$ is arbitrary but there are homomorphisms $A \rightarrow C$ and $B \rightarrow C$. Since such homomorphisms exist if both $A$ and $B$ are bipartite (and $C$ has at least one edge) then cancellation can fail only if $C$ is bipartite and $A$ and $B$ are not both bipartite. Failure of cancellation can thus be divided into two cases, both involving a bipartite factor $C$. On one hand it is possible for cancellation to fail if $A$ and $B$ are both non-bipartite. For example, if $A = K_3$ and $B$ is the path of length two with loops at each end, then $A \times K_2$ and $B \times K_2$ are both isomorphic to the 6-cycle, but $A \not\cong B$.

On the other hand, cancellation can fail if only one of $A$ and $B$ is bipartite. Figures 1(a) and 1(b) show an example. In those figures, $A$ consists of two copies of an edge with loops at both ends, $B$ is the four-cycle, and $C$ is the path of length 2. The figures show that $A \times C \cong B \times C$, but clearly $A \not\cong B$.

![Figure 1(a)](image1a.png)  
![Figure 1(b)](image1b.png)

This note is concerned with the second case. We describe the exact conditions a bipartite graph $B$ must meet in order for $A \times C \cong B \times C$ to imply $A \cong B$. Specifically, we prove that if $B$ and $C$ are both bipartite, then $A \times C \cong B \times C$ necessarily implies that $A \cong B$ if and only if no component of $B$ admits an involution (that is an automorphism of order two) that interchanges its partite sets. Figure 1 can be taken as an illustration of this. The 4-cycle $B$ in Figure 1(b) has an involution that interchanges its partite sets (reflection across the vertical axis) and indeed cancellation fails. Our result will imply that if a bipartite graph $B$ does not have this kind of symmetry (or more precisely if no component of $B$ has such symmetry) then $A \times C \cong B \times C$ will guarantee that $A \cong B$. Conversely, if some component of $B$ has a bipartition-reversing involution, then there is a graph $A$ with $A \times C \cong B \times C$ but $A \not\cong B$. 
The reader is assumed to be familiar with the basic properties of direct products, including Weichsel’s theorem on connectivity. See Chapter 5 of [3] for an excellent survey.

2. Results

In what follows, let $V(K_2) = \{0, 1\}$. For $\varepsilon \in V(K_2)$, set $\overline{\varepsilon} = 1 - \varepsilon$, so $\overline{0} = 0$ and $\overline{1} = 1$. An involution of a graph is an automorphism $\beta$ for which $\beta^2$ is the identity. Recall that if $G$ is a connected non-bipartite graph, then $G \times K_2$ is a connected bipartite graph, and $(g, \varepsilon) \mapsto (g, \overline{\varepsilon})$ is an involution of $G \times K_2$ that interchanges the partite sets $V(G) \times \{0\}$ and $V(G) \times \{1\}$. By contrast, if $G$ is bipartite, then $G \times K_2 \cong 2G$, where $2G$ designates the disjoint union of two copies of $G$. We will need the following lemma. It appeared in [1], but it is included here for completeness.

**Lemma 1.** Suppose $A, B$ and $C$ are graphs and $C$ has at least one edge. Then $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$.

**Proof.** Given digraphs $X$ and $Y$, let $\text{hom}(X, Y)$ be the number of homomorphisms from $X$ to $Y$. We will use the following theorem of Lovász: If $D$ and $D'$ are digraphs, then $D \cong D'$ if and only if $\text{hom}(X, D) = \text{hom}(X, D')$ for all digraphs $X$ ([2], Theorem 2.11). We will also use the fact that $\text{hom}(X, A \times B) = \text{hom}(X, A) \text{hom}(X, B)$ for all digraphs $X, A$ and $B$. ([2], Corollary 2.3).

Identify $A, B, C$ and $K_2$ with their symmetric digraphs (i.e., each edge is replaced with a double arc). If we can show $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$ for the symmetric digraphs, then certainly this holds for the underlying graphs as well.

From $A \times C \cong B \times C$ we get $(A \times K_2) \times C \cong (B \times K_2) \times C$. Let $X$ be a digraph. Then

$$\text{hom}(X, A \times K_2) \text{hom}(X, C) = \text{hom}(X, (A \times K_2) \times C)$$
$$= \text{hom}(X, (B \times K_2) \times C)$$
$$= \text{hom}(X, B \times K_2) \text{hom}(X, C).$$

If $X$ is bipartite (i.e., if its underlying graph is bipartite) then $\text{hom}(X, C) \neq 0$ because the map sending two partite sets to the two endpoints of a double arc of $C$ is a homomorphism. Thus $\text{hom}(X, A \times K_2) = \text{hom}(X, B \times K_2)$. 
On the other hand, if $X$ is not bipartite, then there can be no homomorphism from $X$ to a bipartite graph, and hence $\text{hom}(X, A \times K_2) = 0 = \text{hom}(X, B \times K_2)$. Thus $\text{hom}(X, A \times K_2) = \text{hom}(X, B \times K_2)$ for any $X$, so Lovász’s theorem gives $A \times K_2 \cong B \times K_2$.

We are now in a position to prove our main result.

**Proposition 1.** Suppose $A, B$ and $C$ are graphs for which $B$ and $C$ are bipartite and $C$ has at least one edge. If $A \cong B \times C$ and no component of $B$ admits an involution that interchanges its partite sets, then $A \cong B$.

**Proof.** Let $A$, $B$ and $C$ be as stated. Suppose $A \cong B \times C$, and no component of $B$ admits an involution that interchanges its partite sets. From $A \cong B \times C$, the lemma yields $A \times K_2 \cong B \times K_2$. List the components of $A$ as $A_1, A_2, \ldots A_m$, and those of $B$ as $B_1, B_2, \ldots B_n$, so that $A = \sum_{i=1}^m A_i$ and $B = \sum_{j=1}^n B_j$, where the sums indicate disjoint union. Then

$$A \times K_2 \cong B \times K_2,$$

$$\left( \sum_{i=1}^m A_i \right) \times K_2 \cong \left( \sum_{j=1}^n B_j \right) \times K_2,$$

$$\sum_{i=1}^m (A_i \times K_2) \cong \sum_{j=1}^n 2B_j.$$

From this last equation we see that if $A$ had a component $A_i$ that was not bipartite, then some component $B_j$ of $B$ would be isomorphic to $A_i \times K_2$.

But $A_i \times K_2$ has a bipartition-reversing involution $(a, \bar{a}) \mapsto (a, \bar{a})$, contradicting the fact that no component of $B$ has such an involution. Therefore every component $A_i$ of $A$ is bipartite, so $A$ is bipartite. Then $A \times K_2 \cong B \times K_2$ implies $2A \cong 2B$, whence $A \cong B$.

Conversely, suppose $B$ has a component $B_1$ for which there is an involution $\beta : B_1 \to B_1$ that interchanges the partite sets of $B_1$. We need to produce a graph $A$ with $A \not\cong B$, but $A \times C \cong B \times C$.

Say the partite sets of $B_1$ are $X$ and $Y$, so $\beta(X) = Y$. Define a graph $B'_1$ as $V(B'_1) = V(B_1)$ and $E(B'_1) = \{b\beta(b') : bb' \in E(B_1)\}$. Notice that for each edge $bb'$ of $B_1$, the graph $B'_1$ has edges $b\beta(b')$ and $\beta(b)b'$, and conversely
every edge of $B'_1$ has such a form. It follows that every edge of $B'_1$ has both endpoints in $X$ or both endpoints in $Y$, so $B'_1$ is disconnected. (Example: Let $B_1$ be the graph $B$ in Figure 1(b), and let $\beta$ be reflection across the vertical axis. Then $B'_1$ is the graph $A$ in Figure 1(a).)

Let $A = B'_1 + B_2 + B_3 + \cdots + B_n$. In words, $A$ is identical to $B$ except the component $B_1$ of $B$ is replaced with $B'_1$. Then $A \not\cong B$ because $A$ has more components than $B$.

However, we claim $A \times C \cong B \times C$. To prove this, it suffices to show $B'_1 \times C \cong B_1 \times C$. (For $A$ and $B$ are identical except for $B'_1$ and $B_1$.) Select a bipartition $V(C) = C_0 \cup C_1$ of $C$. Define a map $\theta : B_1 \times C \rightarrow B'_1 \times C$ as

$$\theta(b, c) = \begin{cases} (b, c) & \text{if } c \in C_0, \\ (\beta(b), c) & \text{if } c \in C_1. \end{cases}$$

Certainly this is a bijection of vertex sets. But it is an isomorphism as well, as follows. Suppose $(b, c)(b', c') \in E(B_1 \times C)$. Then $bb' \in E(B_1)$ and $cc' \in E(C)$. We may assume $c \in C_0$ and $c' \in C_1$, so $\theta(b, c)\theta(b', c') = (b, c)(\beta(b'), c')$. But $b\beta(b') \in E(B'_1)$, by definition of $B'_1$, so it follows $\theta(b, c)\theta(b', c') \in E(B'_1 \times C)$. In the other direction, suppose $\theta(b, c)\theta(b', c') \in E(B'_1 \times C)$. From this and by definition of $\theta$, it follows that $cc' \in E(C)$, so we may assume $c \in C_0$ and $c' \in C_1$. Then we have $\theta(b, c)\theta(b', c') = (b, c)(\beta(b'), c') \in E(B'_1 \times C)$. In particular, $b\beta(b') \in E(B'_1)$, and by definition of the edge set of $B'_1$, this means that either $bb' \in E(B_1)$ or $\beta^{-1}(b)\beta(b') \in E(B_1)$. In the latter case, since $\beta$ is an involution we have $\beta(b)\beta(b') \in E(B_1)$, so $bb' \in E(B_1)$. Either way, $bb' \in E(B_1)$, so $(b, c)(b', c') \in E(B_1 \times C)$. Thus $\theta$ is an isomorphism.

Consequently, $A \times C \cong B \times C$, but $A \not\cong B$.

To conclude, we mention one open question suggested by our result. In the introduction we noted that cancellation of $A \times C \cong B \times C$ can fail only if $C$ is bipartite and at least one of $A$ or $B$ is not bipartite. (We assume, as always, that $C$ has at least one edge.) Given that $C$ is bipartite, our result completely characterizes whether or not cancellation holds in the case that $B$ is bipartite. It does not address the situation in which neither $A$ nor $B$ is bipartite. Thus, to complete the picture we would need to understand structural properties of non-bipartite graphs $A$ and $B$ that characterize whether or not cancellation of $A \times C \cong B \times C$ holds.

Here is one perspective on this question. The article [1] introduces an equivalence relation on graphs as $A \sim B$ if and only if $A \times K_2 \cong B \times K_2$. 

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It is proved that if $C$ is bipartite (and has an edge), then $A \times C \cong B \times C$ if and only if $A \sim B$. Let $[A] = \{G \in \Gamma_0 : G \sim A\}$ be the equivalence class containing $A$. Then for bipartite $C$, cancellation in $A \times C \cong B \times C$ holds if and only if the class $[A]$ (hence also $[B]$) contains only one graph. The present note implies that for a bipartite graph $B$, we have $[B] = \{B\}$ if and only if no component of $B$ admits a bipartition-reversing involution. It remains to characterize which classes contain a single non-bipartite graph.

References


