PART I. Prove the following statements.

1. Prove that an integer \( a \) is even if and only if \( a^2 + 2a + 9 \) is odd.

   **Proof.** First we will show that if \( a \) is even, then \( a^2 + 2a + 9 \) is odd. We use direct proof.
   Suppose \( a \) is even. Then \( a = 2k \) for some integer \( k \), and
   \[
   a^2 + 2a + 9 = (2k)^2 + 2(2k) + 9 = 4k^2 + 4k + 8 + 1 = 2(2k^2 + 2k + 4) + 1.
   \]
   This shows that \( a^2 + 2a + 9 \) is twice an integer plus 1, so it is odd.

   Conversely, we will show that if \( a^2 + 2a + 9 \) is odd, then \( a \) is even.
   We use contrapositive proof; that is we will assume \( a \) is not even and show \( a^2 + 2a + 9 \) is not odd.
   Suppose \( a \) is not even, so it is odd, and thus \( a = 2k + 1 \) for some integer \( k \). Then
   \[
   a^2 + 2a + 9 = (2k + 1)^2 + 2(2k + 1) + 9 = 4k^2 + 4k + 1 + 4k + 2 + 9
   = 4k^2 + 8k + 12
   = 2(2k^2 + 4k + 6).
   \]
   This shows that \( a^2 + 2a + 9 \) is twice an integer, so it is even.

   The proof is now complete.

2. Suppose \( A, B \) and \( C \) are nonempty sets. Prove that if \( A \times B \subseteq B \times C \), then \( A \subseteq C \).

   **Proof.** We will use direct proof. Suppose \( A \times B \subseteq B \times C \).

   In what follows we show \( A \subseteq C \).
   Suppose \( a \in A \).
   Since \( B \) is not empty, there is an element \( b \in B \), so \( (a, b) \in A \times B \). (By definition of \( \times \).)
   But since \( A \times B \subseteq B \times C \), it follows that \( (a, b) \in B \times C \). (By definition of \( \subseteq \).)
   In particular, this gives us \( a \in B \), so it now follows that \( (a, a) \in A \times B \). (By definition of \( \times \).)
   But again, since \( A \times B \subseteq B \times C \), it we get \( (a, a) \in A \times C \). (By definition of \( \subseteq \).)
   In particular, this means \( a \in C \). (By definition of \( \times \).)
   We’ve now shown \( a \in A \) implies \( a \in C \), so \( A \subseteq C \).
3. Use induction to prove that \[ 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} .\]

We will prove this with mathematical induction.

(1) When \( n = 1 \) the statement is \( 1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \), which is true.

(2) Now assume the statement is true for some integer \( n = k \geq 1 \), that is assume
\[ 1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 = \frac{k^2(k + 1)^2}{4} . \]

Observe that this implies the statement is true for \( n = k + 1 \), as follows:
\[
\begin{align*}
1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 + (k + 1)^3 &= \frac{k^2(k + 1)^2}{4} + (k + 1)^3 \\
&= \frac{k^2(k + 1)^2}{4} + \frac{4(k + 1)^3}{4} \\
&= \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4} \\
&= \frac{(k + 1)^2(k^2 + 4(k + 1))}{4} \\
&= \frac{(k + 1)^2(k + 2)^2}{4} \\
&= \frac{(k + 1)^2((k + 1) + 1)^2}{4} \\
\end{align*}
\]

Therefore \( 1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 + (k + 1)^3 = \frac{(k + 1)^2((k + 1) + 1)^2}{4} \),

which means the statement is true for \( n = k + 1 \).

This completes the proof by mathematical induction.
4. There exists a set $X$ for which $Z \in X$, $N \in \mathcal{P}(X)$ and $R \in \mathcal{P}(X)$.

**Proof.** Consider the set $X = \{Z\} \cup R$.
(That is, $X$ contains every real number, and it also contains the set of all integers.)
We have $N \subseteq X$ and $R \subseteq X$, and this means $N \in \mathcal{P}(X)$ and $R \in \mathcal{P}(X)$.
Also, we have $Z \in \{Z\}$, so $Z \in \{Z\} \cup R = X$.

5. Use induction to prove that $24|(5^{2n} - 1)$ for every integer $n \geq 0$.

**Proof.** The proof is by mathematical induction.

(1) For $n = 0$, the statement is $24|(5^{2\cdot0} - 1)$. This simplifies to $24|0$, which is true.

(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $24|(5^{2k} - 1)$.
This means $5^{2k} - 1 = 24a$ for some integer $a$, and from this we get $5^{2k} = 24a + 1$.
Now observe that

\[
\begin{align*}
5^{2(k+1)} - 1 &= \\
5^{2k+2} - 1 &= \\
5^25^{2k} - 1 &= \\
5^2(24a + 1) - 1 &= \\
25(24a + 1) - 1 &= \\
25 \cdot 24a + 25 - 1 &= 24(25a + 1)
\end{align*}
\]

This shows $5^{2(k+1)} - 1 = 24(25a + 1)$, which means $24|5^{2(k+1)} - 1$.

This completes the proof by mathematical induction.
PART II. Decide if the following statements are true or false. Prove the true statements; disprove the false ones.

7. If \( A, B \) and \( C \) are sets, then \( A \cup (B - C) = (A \cup B) - (A \cup C) \).

   This is FALSE. Here is a counterexample:
   
   Let \( A = B = C = \{1\} \).
   Then \( A \cup (B - C) = \{1\} \).
   Also \( (A \cup B) - (A \cup C) = \emptyset \).
   This example shows that it is not always true that \( A \cup (B - C) = (A \cup B) - (A \cup C) \).

8. Suppose \( a \) and \( b \) are integres. If \( a|b \) and \( b|a \), then \( a = b \).

   This is FALSE. Here is a counterexample:
   
   Let \( a = 2 \) and \( b = -2 \).
   Then \( a|b \) and \( b|a \), but \( a \neq b \).

9. If \( A, B, C \) are sets and \( A \cap B \cap C = \emptyset \), then \( |A \cup B \cup C| = |A| + |B| + |C| \).

   This is FALSE. Here is a counterexample:
   
   Let \( A = \{1, 2\} \), \( B = \{2, 3\} \) and \( C = \{3, 1\} \).
   Then \( |A \cup B \cup C| = |\{1, 2, 3\}| = 3 \neq 6 = |A| + |B| + |C| \).