1. Suppose \( a, b \in \mathbb{Z} \). Prove that \((a - 3)b^2\) is even if and only if \( a \) is odd or \( b \) is even.

**Proof.** First we will prove that if \((a - 3)b^2\) is even, then \( a \) is odd or \( b \) is even. For this we use contrapositive proof. Suppose it is not the case that \( a \) is odd or \( b \) is even. Then by DeMorgan’s law, \( a \) is even and \( b \) is odd. Thus there are integers \( m \) and \( n \) for which \( a = 2m \) and \( b = 2n + 1 \). Now observe \((a - 3)b^2 = (2m - 3)(2n + 1)^2 = (2m-3)(4n^2 + 4n + 1) = 8mn + 2m - 6n - 3 = 2mn^2 + 4mn + 2m - 6n - 4 + 1 = 2(mn^2 + 4mn + m - 3n - 2) + 1\). This shows \((a - 3)b^2\) is odd, so it’s not even.

Conversely, we need to show that if \( a \) is odd or \( b \) is even, then \((a - 3)b^2\) is even. For this we use direct proof, with cases. **Case 1.** Suppose \( a \) is odd. Then \( a = 2m + 1 \) for some integer \( m \). Thus \((a - 3)b^2 = (2m + 1 - 3)b^2 = (2m - 2)b^2 = 2(m - 1)b^2\). Thus in this case \((a - 3)b^2\) is even.

**Case 2.** Suppose \( b \) is even. Then \( b = 2n \) for some integer \( n \). Thus \((a - 3)b^2 = (a - 3)(2n)^2 = (a - 3)4n^2 = 2(a - 3)2n^2\). Thus in this case \((a - 3)b^2\) is even.

Therefore, in any event, \((a - 3)b^2\) is even.

2. Suppose \( A \) and \( B \) are sets. Prove that \( \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \).

**Proof.** To prove \( \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B) \), we must prove \( \mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B) \) and \( \mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B) \).

First we will show that \( \mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B) \).

Suppose \( X \in \mathcal{P}(A) \cap \mathcal{P}(B) \).

By definition of intersection, this means \( X \in \mathcal{P}(A) \) and \( X \in \mathcal{P}(B) \).

By the definition of power sets, this means \( X \subseteq A \) and \( X \subseteq B \).

Thus, any element \( x \in X \) is in both \( A \) and \( B \), so \( x \in A \cap B \). Hence \( X \subseteq A \cap B \), which means \( X \in \mathcal{P}(A \cap B) \).

We have seen that \( X \in \mathcal{P}(A) \cap \mathcal{P}(B) \) implies \( X \in \mathcal{P}(A \cap B) \), so \( \mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B) \).

Next we will show that \( \mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B) \).

Suppose \( X \in \mathcal{P}(A \cap B) \).

By definition of the power set, this means \( X \subseteq A \cap B \).

Thus any element \( x \in X \) is in \( A \cap B \), so \( x \in A \) and \( x \in B \). Hence \( X \subseteq A \) and \( X \subseteq B \).

Thus \( X \in \mathcal{P}(A) \) and \( X \in \mathcal{P}(B) \), so \( X \in \mathcal{P}(A) \cap \mathcal{P}(B) \), by definition of intersection.

We have seen that \( X \in \mathcal{P}(A \cap B) \) implies \( X \in \mathcal{P}(A) \cap \mathcal{P}(B) \), so \( \mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B) \).

The previous two paragraphs imply \( \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B) \).
3. Suppose $A, B$ and $C$ are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

**Proof.** This is a conditional statement, and we’ll prove it with direct proof. Suppose $B \subseteq C$. (Now we need to prove $A \times B \subseteq A \times C$.)

Suppose $(a, b) \in A \times B$. Then by definition of the Cartesian product we have $a \in A$ and $b \in B$. But since $b \in B$ and $B \subseteq C$, we have $b \in C$. Since $a \in A$ and $b \in C$, it follows that $(a, b) \in A \times C$. Now we’ve shown $(a, b) \in A \times B$ implies $(a, b) \in A \times C$, so $A \times B \subseteq A \times C$.

In summary, we’ve shown that if $B \subseteq C$, then $A \times B \subseteq A \times C$. This completes the proof.

4. Prove by induction: If $n \in \mathbb{N}$, then $6|(n^3 - n)$.

**Proof.** The proof is by mathematical induction.

(a) When $n = 1$, the statement is $6|(1^3 - 1)$, or $6|0$, which is true.

(b) Now assume the statement is true for some integer $n = k \geq 0$, that is assume $6|(k^3 - k)$. This means $k^3 - k = 6a$ for some integer $a$. We need to show that $6|((k + 1)^3 - (k + 1))$. Observe that

\[
(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1
\]
\[
= (k^3 - k) + 3k^2 + 3k
\]
\[
= 6a + 3k^2 + 3k
\]
\[
= 6a + 3k(k + 1)
\]

Thus we have deduced $(k + 1)^3 - (k + 1) = 6a + 3k(k + 1)$. Since one of $k$ or $(k + 1)$ must be even, it follows that $k(k + 1)$ is even, so $k(k + 1) = 2b$ for some integer $b$. Consequently $(k + 1)^3 - (k + 1) = 6a + 3k(k + 1) = 6a + 3(2b) = 6(a + b)$. Since $(k + 1)^3 - (k + 1) = 6(a + b)$ it follows that $6|((k + 1)^3 - (k + 1))$.

Thus the result follows by mathematical induction.
5. Let $A$ and $B$ be sets. If $A - B = B - A$, then $A - B = \emptyset$.

This is TRUE.

**Proof.** Suppose for the sake of contradiction that $A - B = B - A$ but $A - B \neq \emptyset$.
Now since $A - B \neq \emptyset$, then there must be some $a \in A - B$.
And since $a \in A - B = \{x \in A : x \notin B\}$, it follows that $a \in A$ but $a \notin B$.
But also $a \in A - B = B - A = \{x \in B : x \notin A\}$, which means $x \notin A$.
Thus $a \in A$ and $a \notin A$, which is a contradiction. $lacksquare$

6. For every two sets $A$ and $B$, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

This is FALSE. Here is a counterexample.

Let $A = \{1\}$ and $B = \{2\}$.
Then $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
Also $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$.
Thus we see that $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

7. Suppose $A, B, C$ and $D$ are sets. If $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

This is FALSE. Here is a counterexample.
Suppose $A = \{1\}, B = \emptyset, C = \{2\}$ and $D = \{3\}$.
Then $A \times B = \emptyset \subseteq C \times D$, but $A \not\subseteq C$, so it is not true that $A \subseteq C$ and $B \subseteq D$. 