1. Suppose $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.
(Suggestion: Try direct proof.)

**Proof.** (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.

By definition of congruence modulo $n$, this means $n|(a - b)$ and $n|(c - d)$.

By definition of divisibility, $a - b = nk$ and $c - d = n\ell$ for some $k, \ell \in \mathbb{Z}$.

Therefore we have $a = b + nk$ and $c = d + n\ell$. Consequently,

\[
\begin{align*}
ac &= (b + nk)(d + n\ell) \\
ac &= bd + bnl + nkd + n^2k\ell \\
ac - bd &= bnl + nkd + n^2k\ell \\
ac - bd &= n(b\ell + kd + n\ell).
\end{align*}
\]

Since $b\ell + kd + n\ell \in \mathbb{Z}$, it follows from the above equation that $n|(ac - bd)$.

This means that $ac \equiv bd \pmod{n}$.

2. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then both $a$ and $b$ are odd.
(Suggestion: Try contrapositive proof.)

**Proof.** (Contrapositive) Suppose it is not the case that $a$ and $b$ are odd.

Then, by DeMorgan’s Law, $a$ is even or $b$ is even. Let us look at these cases separately.

**Case 1.** Suppose $a$ is even. Then $a = 2c$ for some integer $c$.

Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

**Case 2.** Suppose $b$ is even. Then $b = 2c$ for some integer $c$.

Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c))$, which is even.

Thus in either case $a^2(b^2 - 2b)$ is even, so it is not odd.

*(Note: A third case where both $a$ and $b$ are even is not necessary. In that case $a$ is even, a scenario addressed in Case 1.)*
3. Prove: If \(a, b \in \mathbb{Z}\), then \(a^2 - 4b - 2 \neq 0\).

(Suggestion: Contradiction may be easiest.)

**Proof.** Suppose for the sake of contradiction that \(a, b \in \mathbb{Z}\) but \(a^2 - 4b - 2 = 0\). Then we have \(a^2 = 4b + 2 = 2(2b + 1)\), which means \(a^2\) is even. Therefore \(a\) is even also, so \(a = 2c\) for some integer \(c\). Plugging this back into \(a^2 - 4b - 3 = 0\) gives us

\[
\begin{align*}
(2c)^2 - 4b - 2 &= 0 \\
4c^2 - 4b - 2 &= 0 \\
4c^2 - 4b &= 2 \\
2c^2 - 2b &= 1 \\
2(c^2 - b) &= 1 
\end{align*}
\]

From this last equation, we conclude that 1 is an even number, a contradiction.

4. Suppose \(a, b, c \in \mathbb{Z}\), and \(a \neq 0\). Prove the following statement: If \(a \nmid bc\), then \(a \nmid b\) and \(a \nmid c\).

**Proof.** (Contrapositive) Assume that it is not true that \(a \nmid b\) and \(a \nmid c\). Then \(a \mid b\) or \(a \mid c\). Thus \(b = ak\) or \(c = ak\) for some \(k \in \mathbb{Z}\). Consider these cases separately.

*Case 1.* If \(b = ak\), then multiply both sides by \(c\) to get \(bc = a(kc)\), which means \(a \mid bc\).

*Case 2.* If \(c = ak\), then multiply both sides by \(b\) to get \(bc = a(kb)\), which means \(a \mid bc\).

Thus, in either case \(a \mid bc\), so it is not true that \(a \nmid b\).