Chapter 9 Cancellation (Continued)

Today's Goal

Theorem 9.10 (Lovász) Let $A, B, C \in \Gamma_0$.
If $A \times C \cong B \times C$ and C has an odd cycle, then $A \cong B$

Ingredients

Theorem 5.9 (Weichsel's Theorem)

$$\begin{pmatrix}\text{connected}\\ \text{non-bipartite}\end{pmatrix} \times \begin{pmatrix}\text{connected}\\ \text{non-bipartite}\end{pmatrix} = \begin{pmatrix}\text{connected}\\ \text{non-bipartite}\end{pmatrix}$$

$$\begin{pmatrix}\text{connected}\\ \text{non-bipartite}\end{pmatrix} \times \begin{pmatrix}\text{connected}\\ \text{bipartite}\end{pmatrix} = \begin{pmatrix}\text{connected}\\ \text{bipartite}\end{pmatrix}$$

$$\begin{pmatrix}\text{connected}\\ \text{bipartite}\end{pmatrix} \times \begin{pmatrix}\text{connected}\\ \text{bipartite}\end{pmatrix} = \begin{pmatrix}\text{two bipartite}\\ \text{components}\end{pmatrix}$$

Theorem 8.17 ~~Graphs~~ Connected non-bipartite graphs in Γ_0 factor uniquely into primes over $\times$.

Proposition 9.6 If $A \times C \cong B \times C$ and C has a loop, then $A \cong B$.

Proposition 9.7 If $A \times C \cong B \times C$ and there are homomorphisms $A \to C$ and $B \to C$, then $A \cong B$

Corollary 9.8 If $A \times C \cong B \times C$ and A and B are bipartite, and C has an edge, then $A \cong B$.

Proposition 9.8 If $A \times C \cong B \times C$ and there is a homomorphism $D \to C$, then $A \times D \cong B \times D$.

<u>Theorem A</u> Let A and B be connected. If C has an odd cycle, then $A \times C \cong B \times C \Rightarrow A \cong B$.

<u>Proof</u> Suppose $A \times C \cong B \times C$

<u>CASE 1</u> Suppose C has a loop. Then $A \cong B$ by Prop. 9.6

<u>CASE 2</u> C has no loops. Then C has an odd cycle $C_p \subseteq C$ for $p \geq 3$. So there is a homomorphism $C_p \to C$ (inclusion). Hence $A \times C_p \cong B \times C_p$ ~~by~~ Prop. 9.9

<u>CASE 2A</u> Suppose both A and B are bipartite. Then

$A \cong B$ by Corollary 9.8.

<u>CASE 2B</u> Suppose not both A and B are bipartite. Then they are <u>both</u> <u>non-bipartite</u>, otherwise

$$\underset{\text{bipartite}}{\underset{\uparrow}{A \times C_p}} \cong \underset{\text{not bipartite}}{\underset{\uparrow}{B \times C_p}}$$

By Weichsel's theorem. Now we have

$$A \times C_p \cong B \times C_p \quad \text{(connected non-bipartite)}$$

By unique prime factorization over $\times$, A & B have same prime factorization, so $A \cong B$. ∎

<u>Corollary</u> If $A \times C \cong B \times C$ and C has odd cycle, then $A \cong B$.

<u>Proof</u> As before, if C has a loop then $A \cong B$. Otherwise $A \times C_p \cong B \times C_p$ for some odd cycle C_p.

Write $A = A_1 + \cdots + A_k$ and $B = B_1 + \cdots + B_\ell$ as disjoint unions of their components.

$$A \times C_p \cong B \times C_p$$

$$(A_1 + \cdots + A_k) \times C_p \cong (B_1 + \cdots + B_\ell) \times C_p$$

$$A_1 \times C_p + \cdots + A_k \times C_p \cong B_1 \times C_p + \cdots + B_\ell \times C_p.$$

Then $k = \ell$ and WLOG say $A_i \times C_p \cong B_i \times C_p$ for $1 \leq i \leq k$.

Then $A_i \cong B_i$ $\forall i$ by Theorem [illegible] So $A \cong B$.

§9.3 Antiautomorphisms

$\{ K_2 = \overset{0}{\circ}\!\!-\!\!\overset{1}{\circ} \}$

__Recall__ $A \times C \cong B \times C \Rightarrow A \cong B$ provided C has odd cycle.

Cancellation only fails if C is bipartite.

__Goal__ Let $A \in \Gamma_0$, let C be bipartite. Find all solutions X to $A \times C \cong X \times C$.

There are homomorphisms $K_2 \to C$ and $C \to K_2$, so

$$A \times C \cong X \times C \Rightarrow A \times K_2 \cong X \times K_2 \Rightarrow A \times C \cong X \times C.$$

so $A \times C \cong X \times C \iff A \times K_2 \cong X \times K_2$

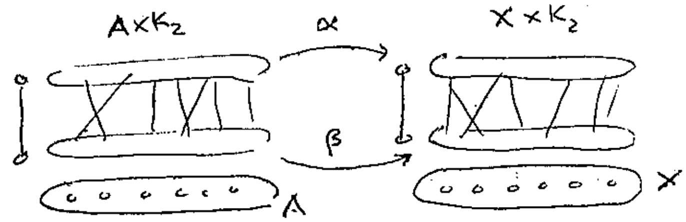
i.e. Solutions to $A \times C \cong X \times C$ and $A \times K_2 \cong X \times K_2$ are identical.

__Goal Restated__ Let $A \in \Gamma_0$. Find all solutions to $A \times K_2 \cong X \times K_2$.

__Recipe for finding a solution X.__

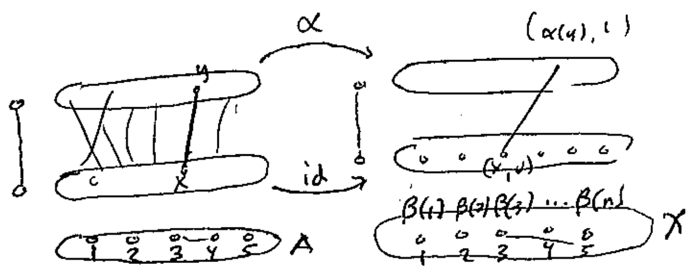
Take isomorphism $\varphi : A \times K_2 \to X \times K_2$

$$\varphi(x, \varepsilon) = \begin{cases} (\alpha(x), \varepsilon) & \text{if } \varepsilon = 1 \\ (\beta(x), \varepsilon) & \text{if } \varepsilon = 0 \end{cases}$$



Relabel vertices of X

$$\varphi(x, \varepsilon) = \begin{cases} (\alpha(x), \varepsilon) & \text{if } \varepsilon = 1 \\ (x, \varepsilon) & \text{if } \varepsilon = 0 \end{cases}$$



$$\begin{aligned} xy \in E(A) &\iff (x,0)(y,1) \in E(A \times K_2) \\ &\iff (x,0)(\alpha(y),1) \in E(X \times K_2) \\ &\iff x\alpha(y) \in E(X) \end{aligned}$$

$xy \in E(A) \iff x\alpha(y) \in E(X)$

$x\alpha^{-1}(y) \in E(A) \iff xy \in E(X)$

$$\begin{aligned} xy \in E(A) &\iff x\alpha(y) \in E(X) \\ &\iff \alpha(y)\,x \in E(X) \\ &\iff \alpha(y)\alpha^{-1}(x) \in E(X) \\ &\iff \alpha^{-1}(x)\alpha(y) \in E(X) \end{aligned}$$

$V(X) = V(A)$

$E(X) = \{ x\alpha(y) \mid xy \in E(A) \}$ where $\alpha : V(A) \to V(A)$ satisfies $\left(xy \in E(A) \Updownarrow \alpha^{-1}(x)\alpha(y) \in E(\right.$

Definitions An <u>antiautomorphism</u> of $A \in \Gamma_0$ is a bijection $\alpha : V(A) \to V(A)$ for which $xy \in E(A) \iff \alpha^{-1}(x)\alpha(y) \in E(A)$ for all $xy \in E(A)$.

Set of all antiautomorphisms of A is denoted $\mathrm{Ant}(A)$.

If $\alpha \in \mathrm{Ant}(A)$, define $A^{\alpha} \in \Gamma_0$ as

$V(A^{\alpha}) = V(A)$

$E(A^{\alpha}) = \{ x\alpha(y) \mid xy \in E(A) \}$

<u>Theorem 9.12</u> The solutions of $A \times C \cong X \times C$ (C bipartite) are $\{ X = A^{\alpha} \mid \alpha \in \mathrm{Ant}(A) \}$

<u>Note</u> This gives all solutions, though some may be repeated
See §9.3 for details.

$\{\alpha^{-1} = \alpha\}$

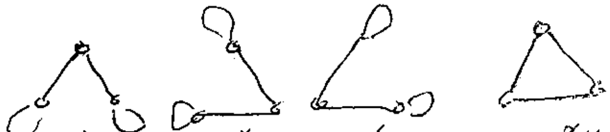
<u>Observation</u> If $\alpha \in \mathrm{Aut}(A)$ and $\alpha^2 = id$ Then $\alpha \in \mathrm{Ant}(A)$

$xy \in E(A) \iff \alpha(x)\alpha(y) \in E(A) \iff \alpha^{-1}(x)\alpha(y) \in E(A)$

$\mathrm{Ant}\Big($  $\Big) = \Big\{$  $\Big\}$

$\uparrow$
A

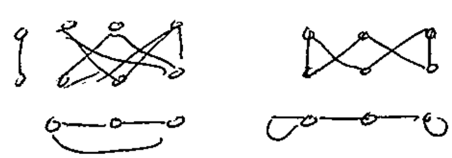
α_1 α_2 α_3 α_4



A^{α_1} A^{α_2} A^{α_3} A^{α_4}

Solutions of $A \times K_2 \cong X \times K_2$ are

$X = A^{\alpha_1} =$ 



$X = A^{\alpha_4} =$ 

$A \times K_2 \cong A^{\alpha_1} \times K_2$