Test One | Advanced Linear Algebra | September 27, 2018

MATH 610

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1. Suppose $T : V \to W$ is a linear transformation. Prove that the range of T is a subspace of W.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation for which $T^2 = T$.

   Show that there is a basis $\mathscr{B}$ of $\mathbb{R}^2$ for which $[T]_{\mathscr{B}}$ is one of the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. Suppose $T : V \to V$ is a linear operator on a 3-dimensional vector space V.

Suppose there is a vector $\alpha \in V$ for which $T^2(\alpha) \neq 0$ but $T^3(\alpha) = 0$.

(a) Show that the set $\mathscr{B} = \{\, \alpha,\ T(\alpha),\ T^2(\alpha) \,\}$ is a basis for V.

(b) Find the matrix of T relative to $\mathscr{B}$, that is, find $[T]_{\mathscr{B}}$.

4. Consider the basis $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Find the dual basis $\mathscr{B}^*$.

5. Suppose a vector space V has basis $\mathscr{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and dual basis $\mathscr{B}^* = \{f_1, f_2, \ldots, f_n\}$.

   Let $\alpha \in V$. Derive the formula $\alpha = \sum_{i=1}^{n} f_i(\alpha)\beta_i$.

6. State the definition of the transpose T^t of a linear transformation $T : V \to W$.

7. Suppose V is the space of all polynomials with coefficients in $\mathbb{R}$, and let $D : V \to V$ be the differentiation operator. (That is, $D(f)$ is the derivative of f.)

   (a) Describe the null space of D^t.

   (b) Let $f \in V^*$ be defined as $f(p) = \int_0^1 p(x)dx$. Find $D^t(f)$. That is, for any $p \in V$, give a formula for $D^t(f)(p)$.