Section 4.3 Polynomial Rings that are UFDs

Recall:
1. \( R[x_1, x_2, \ldots, x_n] = R[x_i, x_j, \ldots, x_k][x_n] \)
2. Any I.D. \( R \) has a field of fractions \( F = \{ \frac{a}{b} | a, b \in R, b \neq 0 \} \), \( R \subseteq F \)
3. Corollary 2: Given \( I \subseteq R \), \( R[x]/(I) \cong R/I[x] \)
   - If \( I \) prime in \( R \), then \( (I) \) prime in \( R[x] \)

We've seen how properties of \( R \) influence properties of \( R[x] \).

1. \( R \) is a field \( \Rightarrow \) \( R[x] \) is an ED, PID, UFD.
2. \( R \) is an I.D. \( \iff \) \( R[x] \) is an I.D.
   
   \( \iff R[x_i, x_j, \ldots, x_k] \) is an I.D.

Today's Goal:

**Theorem 7** \( R \) is a UFD \( \iff \) \( R[x] \) is a UFD

**Corollary 8** \( R \) is a UFD \( \iff \) \( R[x_1, x_2, x_3, \ldots, x_n] \) is a UFD.

The following question is a key to establishing these results. It is answered affirmatively by the so-called Gauss Lemma.

**Question**: If \( f(x) \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x] \) factors in \( \mathbb{Q}[x] \), does it factor in \( \mathbb{Z}[x] \)?
- If \( f(x) \in R[x] \subseteq F[x] \) factors in \( F[x] \), does it factor in \( R[x] \)?

**Proposition 5** Gauss' Lemma

Let \( R \) be an I.D. with field of fractions \( F \) (\( \subseteq \mathbb{Q} \)).

If \( p(x) \in R[x] \) is reducible in \( F[x] \), then it's reducible in \( R[x] \).

Specifically, if \( p(x) = \frac{A(x)B(x)}{R[x]} \), \( \frac{A(x)B(x)}{F[x]} \)

then \( \exists r, s \in F \) such that

\[ p(x) = \frac{rA(x)sB(x)}{R[x]} \]

[Note: necessarily \( rs = 1 \)]
Proof (outline)

Suppose \( p(x) = \frac{A(x)}{\text{R}[x]} \cdot \frac{B(x)}{\text{F}[x]} \).

Then \( \text{d} p(x) = \frac{\text{d} A(x)}{\text{R}[x]} \cdot \frac{e B(x)}{\text{R}[x]} \) for some \( d, e \in \text{R} \).

So \( p_1 p_2 p_3 \ldots p_k p(x) = \frac{d A(x)}{\text{R}[x]} \cdot e B(x) \) \( \ldots \) prime factoring of \( d e \).

Thus \((p_i) \subseteq \text{R}[x]\) is prime ideal in \( \text{R} \).

Corollary 2: \( \text{R}[x] / (p_i) = \text{R}[x] / p_i \text{R}[x] \cong \text{R}/(p_i) [x] \).

Then \( \text{R} / (p_i) \) is ID, \( \Rightarrow \) \( \text{R}[x] / p_i \text{R}[x] \) is ID in \( \text{R}[x] / p_i \text{R}[x] \).

Note: \( \frac{\text{d} A(x)}{\text{R}[x]} \cdot e B(x) + p_i \text{R}[x] = 0 + p_i \text{R}[x] \) in \( \text{R}[x] / p_i \text{R}[x] \).

Say \( \frac{\text{d} A(x)}{\text{R}[x]} = 0 \). \( \Rightarrow \) \( \text{d} A(x) \in p_i \text{R}[x] \). \( \Rightarrow \) \( \frac{1}{p_i} \frac{\text{d} A(x)}{\text{R}[x]} \).

\( \ldots \) \( p_2 p_3 \ldots p_k p(x) = \frac{1}{p_i} \frac{\text{d} A(x)}{\text{R}[x]} \cdot e B(x) \) \( \ldots \) \( \text{R}[x] \).

Continue process with \( p_2 \) instead of \( p_i \), etc.

Get: \( p(x) = \frac{r A(x)}{\text{R}[x]} \cdot \frac{SB(x)}{\text{R}[x]} \).

\( \square \)
Theorem 7  \( R \) is UFD \( \iff \mathbb{R}[x] \) is UFD

Proof  \( (\Leftarrow) \) Trivial because \( \mathbb{R} \subseteq \mathbb{R}[x] \).

\( (\Rightarrow) \) Basic Idea: \( \mathbb{R}[x] \subseteq \mathbb{F}[x] \)

Suppose \( p(x) \in \mathbb{R}[x] \). Need to show \( p(x) \) factors uniquely.

Note that \( p(x) \in \mathbb{F}[x] \) (UFD)

Unique factorization in \( \mathbb{F}[x] \):

\[
p(x) = q_1(x) q_2(x) \ldots q_k(x),
\]

Now use Gauss' Lemma to connect this to a unique factorization in \( \mathbb{R}[x] \).

Corollary 8  \( R \) is UFD \( \iff \mathbb{R}[x_1] \) is UFD.

Proof  Follows from Theorem 7 and \( \mathbb{R}[x_1, x_2, \ldots, x_n] = \mathbb{R}[x_1, x_2, \ldots, x_{n-1}][x_n] \).