Section 10.3 Generation of Modules, Direct Sums, Free Modules.

In what follows we compare structures in an $R$-module $M$ to vector spaces over $R$. All the "new" terms introduced here are entirely parallel to old ideas involving vector spaces.

**VECTOR SPACES OVER $R$**

**Sum of subspaces**

$V_1 + V_2 = \{ a_1 + a_2 \mid a_1 \in V_1, a_2 \in V_2 \}$

**Span of a set $A = \{ a_1, a_2, \ldots, a_k \}$**

$\text{Span}(A) = \{ \sum r_i a_i \mid r_i \in R, a_i \in A \}$

**Finite dimensional space**

If $A$ is finite, then $\text{Span}(A)$ is a finite dimensional vector space.

**One dimensional space**

A subspace $V$ is 1-D if $V = \text{Span}\{a\}$ for some vector $a$.

**R-MODULES $M$**

**Sum of sub-modules**

If $N_1, N_2, \ldots, N_k$ are submodules of $M$, then their sum is

$N_1 + N_2 + \ldots + N_k = \{ a_1 + a_2 + \ldots + a_k \mid a_i \in N_i, 1 \leq i \leq k \}$

**Submodule generated by $A \subset M$.**

If $A \subset M$, the submodule generated by $A$ is

$RA = \{ \sum r_i a_i \mid r_i \in R, a_i \in A \}$

If $N = RA$, we say $N$ is generated by $A$. $A$ is a set of generators for $N$.

**$N \subset M$ is finitely generated if $N = RA$ for some finite set $A$. Similarly, $M$ is finitely generated if $M = RA$ for some finite $A \subset M$.**

**$N \subset M$ is cyclic if there exists an $a \in M$ such that $N = Ra = \{ ra \mid r \in R \}$.**
**Direct Sums and Direct Products**

**Definition** If $M_1, M_2, \ldots, M_k$ are $R$-modules, their direct product is:

$$M_1 \times M_2 \times \cdots \times M_k = \{m_1, m_2, \ldots, m_k \mid m_i \in M_i \text{ } \forall i\}.$$

This is an abelian group and an $R$-module under addition:

$$r(m_1, m_2, \ldots, m_k) = (rm_1, rm_2, \ldots, rm_k).$$

**Alternate Notation:**

$$M_1 \times M_2 \times \cdots \times M_k = M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

`direct product` `direct sum`.

These are the same if $k$ is finite. The difference emerges when there are an $\infty$ number of factors.

**Direct Product**

$$M_1 \times M_2 \times M_3 \times \cdots = \{m_1, m_2, m_3, \ldots \mid m_i \in M_i \text{ } \forall i\}.$$

**Direct Sum**

$$M_1 \oplus M_2 \oplus M_3 \oplus \cdots = \{(m_1, m_2, m_3, \ldots) \mid m_i \in M_i \text{ and all but finitely many } m_i = 0\}.$$

Thus,

$$\bigoplus_{i=1}^{\infty} M_i \subseteq \prod_{i=1}^{\infty} M_i.$$

$$\bigoplus_{i \in I} M_i \subseteq \prod_{i \in I} M_i.$$
A decomposition theorem

For the next result, keep the following vector space picture in mind. The proof mirrors the (almost obvious) vector space setting.

\[ \mathbb{R}^3 \cong V_1 \times V_2 = V_1 \oplus V_2 \]

\[ V_1 \cap V_2 = \{0\} \]

Any \( z \in \mathbb{R}^3 \) has a unique expression

\[ z = a_1 + \alpha_1 \quad a_1 \in V_1, \quad \alpha_1 \in V_2 \]

Many ways to write \( z \in \mathbb{R}^3 \)

Proposition 5: Suppose \( N_1, N_2, \ldots, N_k \) are submodules of the \( R \)-module \( M \). Then the following are equivalent:

1. \( \pi : N_1 \times N_2 \times \cdots \times N_k \to N_1 + N_2 + \cdots + N_k \leq M \) is a submodule isomorphism, where \( \pi(q_1, q_2, \ldots, q_k) = a_1 + a_2 + \cdots + a_k \)

2. \( (N_1 + N_2 + \cdots + N_{j-1} + N_j + \cdots + N_k) \cap N_j = \{0\} \quad \forall j \)

3. Each \( z \in N_1 + N_2 + \cdots + N_k \) has a unique expression

\[ z = a_1 + \alpha_1 + \cdots + a_k \quad a_i \in N_i \]

Example: \( \mathbb{Z} \)-module \( M = \mathbb{Z}/6\mathbb{Z} \)

\[ N_1 = \langle 0, 2, 4 \rangle = \mathbb{Z} 2 = \text{span} \{2\} \]

\[ N_2 = \langle 0, 3 \rangle = \mathbb{Z} 3 = \text{span} \{3\} \]

\[ N_1 + N_2 = \langle 0, 1, 2, 3, 4, 5 \rangle = M \]

Since \( N_1 \cap N_2 = \{0\} \)

\[ N_1 \times N_2 \cong N_1 + N_2 = M \]

\[ (x, y) \mapsto x + y \]

Note: This is not like a vector space in that \( \text{span} \{2\} \neq \mathbb{Z} \), etc.

We next introduce the notion of free modules, for which such "one-dimensional" submodules are isomorphic to \( \mathbb{R} \).
Definition. A module $F$ is free on a set $A$ if for every $x \in F$, there exist unique non-zero $r_1, r_2, \ldots, r_n \in R$ and $a_1, a_2, \ldots, a_n \in A$ for which $x = \sum_{i=1}^{n} r_i a_i$. The rank of $F$ over $A$ is denoted by $\text{rank}_A F$.

Example. \[ R^n = R \times R \times \cdots \times R \quad A = \{(100, \ldots), (010, \ldots), (001, \ldots), (0, 0, \ldots)\} \]

Theorem 6. For any set $A$, there is a free $R$-module $F(A)$ on $A$. Moreover, $F(A)$ satisfies the following universal property:

For any $R$-module $M$ and any map $\varphi : A \to M$, there is a unique $R$-module homomorphism $\Phi : F(A) \to M$ with $\Phi(a_i) = \varphi(a_i)$ such that for all $a \in A$.

If $A$ is finite, i.e., $A = \{a_1, a_2, \ldots, a_n\}$, then $F(A) = R a_1 \oplus R a_2 \oplus \cdots \oplus R a_n = R^n$. 

\[ F(A) = \sum_{\varphi \text{ a function } A \to R} \varphi \]

\[ A \xrightarrow{\text{incl}} F(A) \]

\[ M \xrightarrow{\Phi} \]