Section 2.4 Subgroups Generated by Subsets.

Goal Answer the following question:
We know \( \langle a \rangle = \{ a^n | n \in \mathbb{Z} \} = \langle \{ a^3 \} \rangle \leq G \)
What is \( \langle A \rangle \), where \( A \leq G \)?

Proposition If \( H \leq G \) and \( K \leq G \) then \( HK \leq G \).

Proposition If \( \{ H_\alpha \}_{\alpha \in I} \) is a collection of subgroups \( H_\alpha \leq G \) for each \( \alpha \in I \), then \( \bigcap_{\alpha \in I} H_\alpha \leq G \).

But keep in mind index set \( I \) could have any cardinality.

Definition Given \( A \leq G \), \( \langle A \rangle = \bigcup_{H \leq G} H \leq G \)

Example \( G = \mathbb{Z} \), \( A = \{ 7\mathbb{Z}, 36, 24 \} \)
\( \langle A \rangle = \mathbb{Z} \cap 4\mathbb{Z} \cap 3\mathbb{Z} \cap 6\mathbb{Z} \cap 12\mathbb{Z} \cap \mathbb{Z} = 12\mathbb{Z} \)

Example \( G = \mathbb{R}^\times \), \( A = \{ \pi, \sqrt{2}, 7, e, \sqrt{11} \} \)
\( \langle A \rangle = \{ \} \)

Even though the definition defines \( \langle A \rangle \) unambiguously, it's hard to say exactly what \( \langle A \rangle \) is.
**Definition** If $A \subseteq G$, then

\[ \overline{A} = \{ a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3} \cdots a_n^{\varepsilon_n} \mid a_i \in A, \varepsilon_i = \pm 1 \} \]

\[ = \{ a_1^{p_1} a_2^{p_2} a_3^{p_3} \cdots a_n^{p_n} \mid a_i \in A, p_i \in \mathbb{Z} \} \]

\[ = \text{finite products of powers of elements of } A. \]

**Proposition 9** \[ \overline{A} = \langle A \rangle \]

\[ \uparrow \quad \uparrow \]

of theoretical use (proofs)

of computational use (proofs)

It's good to have two different ways of looking at the same thing.

**Note:** If $G$ is abelian, and $A = \{a_1, a_2, \ldots, a_k\}$, then

\[ \overline{A} = \{ a_1^{p_1} a_2^{p_2} \cdots a_k^{p_k} \mid k \geq 0, p_i \in \mathbb{Z} \} \]

(in order)

**Convention:** \[ \langle \phi \rangle = \overline{\phi} = \mathbb{Z} \]

**Notation:** \[ \langle a_1, a_2, \ldots, a_n \rangle \]

\[ \langle A \cup B \rangle = \langle A, B \rangle \]

**Example** \[ \langle \pi, \sqrt{2}, 7, e^{\sqrt{11}} \rangle \]

\[ = \{ z \pi + y\sqrt{2} + 27 + u \pi + w \sqrt{11} \mid x, y, z, u, w \in \mathbb{R} \} \]

(proper subgroup of $\mathbb{R}$, not cyclic)
Example: Consider \( \mathbb{Z}/36\mathbb{Z} \).

Divisors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36

Subgroups of \( \mathbb{Z}/36\mathbb{Z} \):

- \( \langle 1 \rangle = \{0, 1, 2, \ldots, 35\} = \mathbb{Z}/36\mathbb{Z} \)
- \( \langle 2 \rangle = \{0, 2, 4, \ldots, 34\} \)
- \( \langle 3 \rangle = \{0, 3, 6, \ldots, 33\} \)
- \( \langle 4 \rangle = \{0, 4, 8, 12, \ldots, 32\} \)
- \( \langle 6 \rangle = \{0, 6, 12, 18, 24, 30\} \)
- \( \langle 9 \rangle = \{0, 9, 18, 27\} \)
- \( \langle 12 \rangle = \{0, 12, 24\} \)
- \( \langle 18 \rangle = \{0, 18\} \)
- \( \langle 36 \rangle = \{0\} \)

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Example: Consider \( Q_8 \).

Subgroups of \( Q_8 \):

- \( Q_8 \)
- \( \langle i \rangle = \{1, i, i^2, i^3, i^4\} = \{1, -1, i, -i\} \)
- \( \langle j \rangle = \{1, -1, i, -i\} \)
- \( \langle k \rangle = \{1, i, -1, -i\} \)
- \( \langle -1 \rangle = \{1, -1\} \)
- \( \langle 1 \rangle = \{1\} \) (Note \( \langle i, j \rangle = Q_8 \), etc)

Text makes the point that subgroup lattices can help in the computation of centralizers and centers.

\[ C_{Q_8}(\langle i \rangle) = \langle i \rangle \] because this is smallest subgroup containing \( i \) that commutes with everything in \( \langle i \rangle \).

Also \( Z(Q_8) = \langle -1 \rangle = \{1, -1\} \).