Section 1.7 Continued

Example of an Action

\[ G = \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \text{ acts on } A = \left\{ \begin{bmatrix} \ast & \ast \\ \circ & \circ \end{bmatrix}, \begin{bmatrix} \circ & \ast \\ \circ & \ast \end{bmatrix} \right\} \]

(Entries of vectors are from \( \mathbb{Z}/2\mathbb{Z} \))

The action is matrix multiplication.

Thus, there is a homomorphism \( \varphi: G \rightarrow S_3 \) where \( \varphi(B) = \sigma_B \).

This map is injective:

Suppose \( B, B' \in G \) and \( \varphi(B) = \varphi(B') \)

\[ \sigma_B = \sigma_{B'} \]

\[ \sigma_B(x) = \sigma_{B'}(x) \quad \forall x \in A \]

\[ BX = B'X \quad \forall x \in A \]

\[ B \begin{bmatrix} \ast \\ \circ \end{bmatrix} = B' \begin{bmatrix} \ast \\ \circ \end{bmatrix} \]

\[ \begin{bmatrix} \ast \\ \circ \end{bmatrix} \quad \text{1st col of } B \]

\[ \begin{bmatrix} \ast \\ \circ \end{bmatrix} \quad \text{1st col of } B' \]

\[ B \begin{bmatrix} \circ \\ \ast \end{bmatrix} = B' \begin{bmatrix} \circ \\ \ast \end{bmatrix} \]

\[ \begin{bmatrix} \circ \\ \ast \end{bmatrix} \quad \text{2nd col of } B \]

\[ \begin{bmatrix} \circ \\ \ast \end{bmatrix} \quad \text{2nd col of } B' \]

\[ \Rightarrow \begin{bmatrix} \ast & \ast \\ \circ & \circ \end{bmatrix} = \begin{bmatrix} \ast & \ast \\ \circ & \circ \end{bmatrix} \quad \text{Therefore } B = B' \]

\[ B \begin{bmatrix} \circ \\ \ast \end{bmatrix} = B' \begin{bmatrix} \circ \\ \ast \end{bmatrix} \]

\[ \begin{bmatrix} \circ \\ \ast \end{bmatrix} \quad \text{2nd col of } B \]

\[ \begin{bmatrix} \circ \\ \ast \end{bmatrix} \quad \text{2nd col of } B' \]

Have shown \( \varphi(B) = \varphi(B') \Rightarrow B = B' \) so \( \varphi \) injective.

Therefore action is faithful.

Note injection \( \varphi: G \rightarrow S_3 \) and \( |G| = |S_3| = 6 \)

so \( \varphi \) is also surjective.

We have a bijective homomorphism \( \varphi: \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \rightarrow S_3 \)

so \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \cong S_3 \).
Definition: A subgroup of a group $G$ is a subset $H \subseteq G$ that is itself a group under $G$'s operation. We write $H \leq G$ to indicate $H$ is a subgroup of $G$.

Subgroup Criteria: Suppose $G$ is a group, $H \leq G$.

Proposition: $H \leq G$ if and only if
1. $e \in H$ (or $H \neq \emptyset$)
2. $x, y \in H \Rightarrow xy \in H$ ($H$ is closed)
3. $x \in H \Rightarrow x^{-1} \in H$ ($H$ is closed under inverse)

Proposition: $H \leq G$ if and only if
1. $H \neq \emptyset$
2. $\forall x, y \in H$, $xy^{-1} \in H$

Proposition: Suppose $H$ is finite. Then $H \leq G$ if and only if
1. $H \neq \emptyset$
2. $x, y \in H \Rightarrow xy \in H$

Reason: Suppose $H$ is finite and closed under multiplication. Take $x \in H$. Look at $\{x, x^2, x^3, \ldots \} \subseteq H$
Must have $x^\theta = x^\varphi$ for some $\theta < \varphi$.
Then $e = x^{\varphi - \theta} \in H$
So $e = (x)(x^{\varphi - \theta - 1})$
Thus $x^{-1} = x^{\varphi - \theta - 1} \in H$. 

Examples (Kernels and Stabilizers of actions)

Suppose $G$ acts on a set $A$.

The kernel of the action is
$$\{ g \in G \mid ga = a \ \forall a \in G \} \leq G$$  (This is a subgroup)

Given $s \in A$, the stabilizer of $s$ is
$$\{ g \in G \mid gs = s \} \leq G$$  (This is a subgroup)

Example

\[\mathbb{Z}\] acts on $A = \{1, 2, 3, 4\}$ by $g \cdot 90^\circ$ clockwise.

Kernel: $\{ g \in \mathbb{Z} \mid ga = a \} = \{ 0, \pm 4, \pm 8, \ldots \} \cong 4\mathbb{Z} \leq \mathbb{Z}

Stabilizers of $s = (11) \quad \{ g \in \mathbb{Z} \mid gs = s \} = 4\mathbb{Z}$

Stabilizer of $s = (90) \quad \{ g \in \mathbb{Z} \mid gs = s \} = \mathbb{Z}$

$\mathbb{Z}$ also acts on $P(A)$. Given $X \in P(A)$, i.e. $X \subseteq A$ we put $g \cdot X = \{ g \cdot x \mid x \in X \}$. (Check that this is an action).

Stabilizer of $X = \{(11), (-1,-1)\}$ is $\{ g \in \mathbb{Z} \mid gX = X \} = 2\mathbb{Z} \leq \mathbb{Z}$.

Example

$G$ acts on itself by conjugation.

Given $g \in G$ and $a \in G$, $g \cdot a = gag^{-1}$.

(Check that this is an action.)

Kernel of action is
$$\{ g \in G \mid ga = a \} = \{ g \in G \mid gag^{-1} = a \ \forall a \in G \}$$
$$= \{ g \in G \mid ga = ag \ \forall a \in G \} \leq G$$

This is the set of all $g \in G$ that commute with everything in $G$. Called the center of $G$. 
Centers, Centralizers, and Normalizers

The center of a group $G$ is the subgroup

\[ Z(G) = \{ g \in G \mid gx = xg \quad \forall x \in G \} \leq G \]

(elements of $G$ that commute with everything)

If $A \leq G$, the centralizer of $A$ is

\[ C_G(A) = \{ g \in G \mid gx = xg \quad \forall x \in A \} \leq G \]

(elements of $G$ that commute with everything in $A$)

Thus $C_G(G) = Z(G)$, $C_G(\mathbb{R}^3) = G$

Exercise: Show $Z(\text{GL}_n(\mathbb{F})) = \left\{ \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \mid d \in \mathbb{F} \right\}$

By Homework § 1.2, 4

\[ Z\left( \text{D}_2(2k) \right) = \{1, r^k \} \]

\[ C_G(A) = \{ g \in G \mid gxg^{-1} = x \quad \forall x \in A \} \leq G \]

The normalizer of $A$ in $G$ is the subgroup

\[ N_G(A) = \{ g \in G \mid gAg^{-1} \in A \quad \forall x \in A \} = \{ g \in G \mid gAg^{-1} = A \} \]

Note: $Z(G) \leq C_G(A) \leq N_G(A)$