Section 1.6 Homomorphisms and Isomorphisms

Given groups $G$ and $H$, there are lots of maps $\varphi: G \to H$. From an algebraic point of view, we are mainly interested in those maps that respect the group structure of $G$ and $H$ in the sense that $\varphi(xy) = \varphi(x)\varphi(y)$.

**Definition** A homomorphism $\varphi: G \to H$ is a map for which $\varphi(xy) = \varphi(x)\varphi(y) \quad \forall \ x, y \in G$.

**Example** $\varphi: GL_n(\mathbb{R}) \to \mathbb{R}^\times = \{r \in \mathbb{R} : r \neq 0\}$

$\varphi(A) = \det(A)$

$\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B)$.

**Example** $\varphi: \mathbb{Z}^x \to \mathbb{R}^\times$ $\varphi(k) = 2^k$

$\varphi(x+y) = 2^{x+y} = 2^x2^y = \varphi(x)\varphi(y)$

**Example** $\varphi: \mathbb{R}^\times \to \mathbb{Z}$ $\varphi(x) = \frac{x}{\ln(2)}$

$\varphi(xy) = \frac{xy}{\ln(2)} = \frac{x}{\ln(2)}\frac{y}{\ln(2)} = \varphi(x)\varphi(y)$

**Example** $\varphi: \mathbb{R}^\times \to \mathbb{Z}$ $\varphi(x) = 1$

Note $\varphi(a^n) = \varphi(aa\ldots a) = \varphi(a)\varphi(a)\ldots = \varphi(a)^n$. $\varphi(a^n) = \varphi(a)$ for any hom. $\varphi(1) = 1$. $\varphi(1) = 1$.

**Definition** An isomorphism $\varphi: G \to H$ is a bijective homomorphism.

If there is an iso $\varphi: G \to H$, we say $G \cong H$.

$\varphi$ "lays $G$ on top of $H"$ so that they match. $H$ is the same group as $G$, with elements $x$, elements of $\varphi(x)$. $\varphi(x)$.
IF $G \cong H$, any algebraic structure that one group has is shared by the other.

Example: $\mathbb{R}^+ \cong \mathbb{R}$ because $\ln : \mathbb{R}^+ \to \mathbb{R}$ is iso.

$\ln(xy) = \ln(x) + \ln(y)$

$\mathbb{R}^\times \cong \mathbb{R}$ because $\mathbb{R}^\times$ has element -1 of order 2, but $\mathbb{R}$ has no such.

Note, however, if $\varphi : G \to H$ is just a homomorphism, then $G$ and $H$ can have different structures.

Example: $\det : GL_2(\mathbb{R}) \to \mathbb{R}^\times$

\[\begin{array}{cc}
\uparrow & \\
\text{non-abelian} & \text{abelian}
\end{array}\]

Defining Homomorphisms between groups with presentations.

Example

$G = \langle a, b | a^6 = 1, b^4 = 1 \rangle = \langle a^{k_1}b^{l_1}, a^{k_2}b^{l_2}, \ldots | 10 \leq k_i, \leq 6, 0 \leq l_i, \leq 4 \rangle$

$H = \langle x, y | x^3 = 1, y^2 = 1, xy = yx \rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Here any relation that holds for $a, b \in G$ also hold in $H$ when $x, y$ are replaced by $a, b$, respectively.

There is a unique homomorphism $\varphi : G \to H$ for which $\varphi(a) = x$, $\varphi(b) = y$.

Define $\varphi(a^k) = x^k = \varphi(a)^k$, $\varphi(b^k) = y^k = \varphi(b)^k$,

$\varphi(a^6) = \varphi(a)^6 = x^6 = (x^3)^2 = 1^2 = 1$

For this to work, we need relation $a^6 = 1$ to hold in $H$ as $x^6 = 1$.

In general, define

$\varphi(a^{k_1}b^{l_1}a^{k_2}b^{l_2} \ldots) = \varphi(a)^{k_1}\varphi(b)^{l_1}\varphi(a)^{k_2}\varphi(b)^{l_2} \ldots$

Note that this is a homomorphism.
Section 1.7  Group Actions

Recall A permutation of a set $A$ is a bijection $\pi: A \to A$.

Example $A = \{\pi, \mu \in S_A\}$ - set of points on plane.

$\pi: A \to A$ is rotation by $90^\circ$ - permutations of $A$.

$\mu: A \to A$ is reflection on $x$-axis.

Notation Let $G$ be a group, $A$ a set. Given $\varphi: G \times A \to A$ we write $\varphi(g, a) = g.a$

Definition An action of a group $G$ on a set $A$ is a map $G \times A \to A$ satisfying:

1. $g^2(g.a) = (g'g).a$, $\forall g, g' \in G$ and $a \in A$
2. e.a = a

Example $G = \mathbb{Z}$, $A = \{\pi, \mu \in S_A\}$

$\varphi(i, a) = i.a =$ rotation by $90^\circ$

e.g. $a$, $a =$ rotation by $0^\circ = 0^\circ$

1. $a =$ rotation by $1.90^\circ = 90^\circ$

2. $a = \pi = 2.90^\circ = 180^\circ$ etc.

This is a group action

1. $i.(i.a) = (\text{rotation of } a \text{ by } 90^\circ \text{ followed by } i) = (i+i).a$

2. $e.a = (\text{rotation of } a \text{ by } 0^\circ = 0^\circ) = a$

Definition If $G$ acts on $A$, the kernel of this action is the set \{ $g \in G \mid g.a = a \ A \ a \in \mathbb{A}$ \}

In the example above, kernel = \{ $0, \pm 4, \pm 8, \pm 12 \ldots \} \subset \mathbb{Z}$.

FACTS (Proved in text) Given action of $G$ on $A$:

If $g \in G$, define $\sigma_g: A \to A$ defined as $\sigma_g(a) = g.a$

(i) For each $g \in G$, $\sigma_g$ is a permutation of $A$.

(ii) The map $G \to S_A$ defined as $g \mapsto \sigma_g$ is a homomorphism
The action of $G$ on $A$ is **faithful** if
$G \rightarrow S_A \ g \mapsto \sigma_g$ is injective
(i.e. if distinct elements of $G$ give distinct permutations of $A$)

In such a case, a copy of $G$ "lives" in $S_A$.

The example above ($\mathbb{Z}$ acting on square) is **not**
faithful. $0, 4 \in \mathbb{Z}$ induce following permutations of $A$:

$0 \mapsto \sigma_0 = \text{rotation by } 0.90^\circ = 0^\circ$

$4 \mapsto \sigma_4 = \text{rotation by } 4.90 = 360^\circ = \text{rotation by } 0^\circ$

Can you think of a faithful action on the square?

One answer:

$\mathbb{Z}/4\mathbb{Z}$ acts on $A = \square$

$g.a = \text{rotation by } g \cdot 90^\circ$. 