Section 1.3  Symmetric Groups

Definition Given set \( \Omega \), the symmetric group on \( \Omega \) is

\[ S_\Omega = \{ f \mid f: \Omega \to \Omega \text{ is a bijection} \} = \{ \text{set of all bijections} \}_\Omega \to \Omega \}

\( S_\Omega \) is a group. Operation is composition

(i) composition is associative
(ii) Identity: \( I: \Omega \to \Omega \), \( I(x) = x \)
(iii) For \( f, g \in S_\Omega \), \( f \circ g \in S_\Omega \) and \( f \circ I = I \circ f = f \)

If \( \Omega = \{1, 2, 3, \ldots, n\} \), we write \( S_\Omega = S_n \) = "Symmetric group of degree \( n \)."

Typical element of \( S_7 \):

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 4 & 7 & 6 \end{pmatrix} \]

Structure of \( \sigma \):

\[ 1 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 6 \rightarrow 7 \]

\( \sigma \) decomposes into three smaller permutations, called cycles

Notation: \( \sigma = (1 \ 3 \ 2 \ 5) \ (4) \ (6 \ 7) \)

Definition A cycle \( (a_1, a_2, a_3, \ldots, a_K) \) is an element of \( S_n \) for which

\[ a_1 \to a_2 \]
\[ a_2 \to a_3 \]
\[ a_3 \to a_4 \]
\[ \vdots \]
\[ a_K \to a_1 \]

and all other elements \( \{1, 2, 3, \ldots, n\} \setminus \{a_1, a_2, a_3, \ldots, a_K\} \) are sent to themselves.
**Fact** Every permutation is a composition of disjoint cycles.

\[ \sigma = (1, 3, 2, 5)(4)(6, 7) \]
\[ = (1, 3, 2, 5)(6, 7) \]
\[ = (3, 2, 5, 1)(7, 6) \]

**Fact** Disjoint cycles commute

\[ (1, 3, 2, 5)(6, 7) = (6, 7)(1, 3, 2, 5) \]

**Composition of cycles**

\[ (1, 4, 2)(1, 3, 2) = (1, 3)(2, 4) \]
\[ (1, 3, 2)(1, 4, 2) = (1, 4)(2, 3) \]

**Inverse of a cycle**

\[ (1, 4, 2)^{-1} = (2, 4, 1) \]

Check:

\[ (1, 4, 2)(2, 4, 1) = (1)(2)(3)(4) = 1 \]

\[ \sigma = (1, 3, 2, 5)(6, 7) \]
\[ \sigma^{-1} = (3, 2, 5, 1)(7, 6) \]

\[ \sigma \sigma^{-1} = (1, 3, 2, 5)(6, 7)(5, 2, 3, 1)(7, 6) = (1)(2)(3) \ldots = 1 \]

**Powers of cycles**

\[ \pi = (1, 2, 3, 4) \]
\[ \pi^2 = (1, 3)(4, 1) \]
\[ \pi^3 = (1, 4, 3, 2) \]
\[ \pi^4 = 1 \]

\[ (1, 3, 2, 4)(5, 6, 7)(6) = \text{lcm}(4, 3, 2) = 12 \]
Examples

$S_2$ = Permutations of $\{1, 2\} = \{1, (12)\}$

$S_2 \cong \mathbb{Z}/2\mathbb{Z}$

$S_3 = \{1, (123), (132), (23), (13), (12)\}$

$= \{1, r, r^2, M_1, M_2, M_3\}$

$S_3 \cong D_6$

$S_4 \not\cong \text{Danything}$