

§8.3 Partition Numbers

A partition of an integer n is a sum $n = \sum_{k=1}^n a_i$ of positive integers. The a_i are called parts of the partition. Partitions can be diagrammed, as follows.

Partitions of $n=5$

$1+1+1+1+1$

1^5



$2+1+1+1$

$2^1 \cdot 1^3$



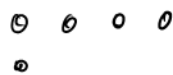
$3+1+1$

$3^1 \cdot 1^2$



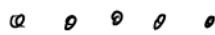
$4+1$

$4^1 \cdot 1^1$



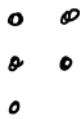
5

5^1



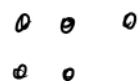
$2+2+1$

$2^2 \cdot 1^1$



$3+2$

$3^1 \cdot 2^1$



Partitions of $n=1$

1

1

Partitions of $n=2$

2

$1 \cdot 1$

$1+1$

$1 \cdot 1$

Partitions of $n=3$

3

$1 \cdot 1 \cdot 1$

$2+1$

$1 \cdot 1 \cdot 1$

$1+1+1$

$1 \cdot 1 \cdot 1$

Partitions of $n=4$

$1+1+1+1$

$1 \cdot 1 \cdot 1 \cdot 1$

$3+1$

$1 \cdot 1 \cdot 1 \cdot 1$

4

$1 \cdot 1 \cdot 1 \cdot 1$

$2+2$

$1 \cdot 1 \cdot 1 \cdot 1$

$2+1+1$

$1 \cdot 1 \cdot 1 \cdot 1$

Let $p_n = (\# \text{ of partitions of } n) = |\{X \mid X \text{ is a partition of } n\}|$

$$p_0 = 1 \quad p_1 = 1 \quad p_2 = 2 \quad p_3 = 3 \quad p_4 = 5 \quad p_5 = 7, \dots$$

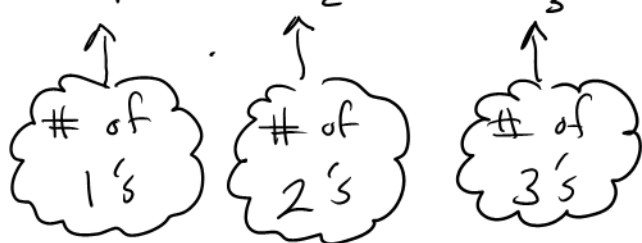
Partition Sequence is

p_0	p_1	p_2	p_3	p_4	p_5	...
1	1	2	3	5	7	...

Unfortunately there is no easy formula for p_n , but there is an interesting generating function for these numbers.

In making a partition of n we choose non-negative integers a_1, a_2, a_3, \dots (possibly equal to zero) with:

$$n = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \dots$$



Generating function for p_n is thus

exponent a_1 is sum of 1's in partition

exponent $2a_2$ is sum of 2's

exponent $3a_3$ is sum of 3's

$$g(x) = (1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)(1+x^3+x^6+x^9+\dots)(1+x^4+x^8+x^{12}+\dots)\dots$$

$$= \frac{1}{1+x} \frac{1}{1+x^2} \frac{1}{1+x^3} \frac{1}{1+x^4} \dots$$

$$= \prod_{k=1}^{\infty} \frac{1}{1+x^k}$$

In multiplying this out before collecting like terms there is one x^n term for each partition of n . Thus coefficient of x^n is p_n

Theorem 8.3.1 Generating function for p_n is $g(x) = \prod_{k=1}^{\infty} \frac{1}{1+x^k}$

Unfortunately the coefficients p_n of x^n are difficult to compute directly because of the infinite product. But we can find any $p_n x^n$ by multiplying the truncated series having all terms of degree $\leq n$. For example, suppose we wanted to find $p_0, p_1, p_2, p_3, \dots, p_{10}$. We look at:

Expand $[(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10})(1+x^2+x^4+x^6+x^8+x^{10})(1+x^3+x^6+x^9)(1+x^4+x^8)(1+x^5+x^{10})(1+x^6)(1+x^7)(1+x^8)(1+x^9)(1+x^{10})]$

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 54x^{11} + 70x^{12} + 91x^{13} + 116x^{14} + 145x^{15} + 181x^{16} + 222x^{17} + 270x^{18} + 325x^{19} + 386x^{20} + 454x^{21} + 529x^{22} + 616x^{23} + 707x^{24} + 805x^{25} + 910x^{26} + 1022x^{27} + 1135x^{28} + 1255x^{29} + 1374x^{30} + 1497x^{31} + 1618x^{32} + 1741x^{33} + 1856x^{34} + 1966x^{35} + 2069x^{36} + 2165x^{37} + 2246x^{38} + 2319x^{39} + 2379x^{40} + 2425x^{41} + 2456x^{42} + 2473x^{43} + 2473x^{44} + 2456x^{45} + 2425x^{46} + 2379x^{47} + 2319x^{48} + 2246x^{49} + 2165x^{50} + 2069x^{51} + 1966x^{52} + 1856x^{53} + 1741x^{54} + 1618x^{55} + 1497x^{56} + 1374x^{57} + 1255x^{58} + 1135x^{59} + 1022x^{60} + 910x^{61} + 805x^{62} + 707x^{63} + 616x^{64} + 529x^{65} + 454x^{66} + 386x^{67} + 325x^{68} + 270x^{69} + 222x^{70} + 181x^{71} + 145x^{72} + 116x^{73} + 91x^{74} + 70x^{75} + 54x^{76} + 42x^{77} + 30x^{78} + 22x^{79} + 15x^{80} + 11x^{81} + 7x^{82} + 5x^{83} + 3x^{84} + 2x^{85} + x^{86} + x^{87}$$

Here we see $p_0=1, p_1=1, p_2=2, p_3=3, p_4=5, p_5=7, p_6=11, p_7=15$

$p_8=22, p_9=30$ and $p_{10}=42$. However the coefficient

of x^{11} does not give p_{11} because the x^{11} term is missing from the first factor. Also note the $1 \cdot x^{87}$ term

This product counts only one partition of 87, namely the one corresponding to the highest powers in the factors, i.e.

$10, 10, 9, 8, 10, 6, 7, 8, 9, 10$, i.e. 10 1's, 5 2's, 3 3's, 2 4's, 2 5's, $16, 17, 18 \notin 19$.

Example Find generating function for $h_n = \left(\begin{array}{l} \# \text{ of partitions of } n \\ \text{into distinct parts} \end{array} \right)$

Ex $6 = 1+5 \quad 6 = 1+5 \quad 6 = 2+4 \quad 6 = 6 \quad 6 = 1+2+3$

$$g(x) = (1+x)(1+x^2)(1+x^3)(1+x^4) \dots$$

$$= \frac{1-x^2}{1-x} \frac{1-(x^2)^2}{1-x^2} \frac{1-(x^3)^2}{1-x^3} \frac{1-(x^4)^2}{1-x^4} \frac{1-(x^5)^2}{1-x^5} \dots$$

$$= \frac{1-x^2}{1-x} \frac{1-x^4}{1-x^2} \frac{1-x^6}{1-x^3} \frac{1-x^8}{1-x^4} \frac{1-x^{10}}{1-x^5} \dots$$

$$= \frac{1}{1-x} \frac{1}{1-x^3} \frac{1}{1-x^5} \dots = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$$

Read The material on orderings of partitions, although we will not need it.

For instance the lexicographic ordering of partitions of n is the usual "dictionary" ordering

Ex 3 3 3 2 1 1 1 1 } partitions of 15
 \leq 3 3 3 3 2 1 0 0

Ordering of partitions of 5

1 1 1 1 1
2 1 1 1
2 2 1
3 1 1
3 2
4 1
5