§8.3 Partition Numbers

A <u>partition</u> of an integer n is a sum $n = \sum_{k=1}^{n} a_i$ of positive integers. The a_i are called <u>parts</u> of the partition. Partitions can be diagramed, as follows.

Partitions of $n = 5$

$1+1+1+1+1$ 1^5

```
o
o
o
o
o
```

$2+1+1+1$ $2^1 \cdot 1^3$

```
o o
o
o
o
```

$3+1+1$ $3^1, 1^2$

```
o o o
o
o
```

$4+1$ $4^1 1^1$

```
o o o o
o
```

5 5^1

```
o o o o o
```

$2+2+1$ $2^2 1^1$

```
o o
o o
o
```

$3+2$ $3^1 2^1$

```
o o o
o o
```

Partitions of $n = 1$

1

```
o
```

Partitions of $n = 2$

2

```
o o
```

$1+1$

```
o
o
```

Partitions of $n = 3$

3

```
o o o
```

$2+1$

```
o o
o
```

$1+1+1$

```
o
o
o
```

Partitions of $n = 4$

$1+1+1+1$

```
o
o
o
o
```

$3+1$

```
o o o
o
```

4

```
o o o o
```

$2+2$

```
o o
o o
```

$2+1+1$

```
o o
o
o
```

Let $p_n = (\#\text{ of partitions of } n) = \left|\{X \mid X \text{ is a partition of } n\}\right|$

$p_0 = 1 \quad p_1 = 1 \quad p_2 = 2 \quad p_3 = 3 \quad p_4 = 5 \quad p_5 = 7, \ldots$

<u>Partition Sequence is</u> $\quad \begin{array}{cccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & \cdots \\ 1 & 1 & 2 & 3 & 5 & 7 & \cdots \end{array}$

Unfortunately there is no easy formula for p_n, but there <u>is</u> an interesting generating function for these numbers.

In making a partition of n, we choose non-negative integers $a_1\ a_2\ a_3 \cdots$ (possibly equal to zero with:

$$n = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots$$

# of 1's

# of 2's

# of 3's

Generating function for p_n is thus

exponent a_1 is sum of 1's in partition

exponent $2a_2$ is sum of 2's

exponent $3a_3$ is sum of 3's

$$g(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)(1+x^4+x^8+x^{12}+\cdots)\cdots$$

$$= \frac{1}{1+x}\ \frac{1}{1+x^2}\ \frac{1}{1+x^3}\ \frac{1}{1-x^4} \cdots$$

$$= \prod_{K=1}^{\infty} \frac{1}{1+x^K}$$

In multiplying this out before collecting like terms there is one x^n term for each partition of n. Thus coefficient of x^n is p_n

Theorem 8.3.1 Generating function for p_n is $g(x) = \prod_{k=1}^{\infty} \frac{1}{1+x^k}$

Unfortunately the coefficients p_n of x^n are difficult to compute directly because of the infinite product. But we can find any $p_n x^n$ by multiplying the truncated series having all terms of degree $\le n$. For example, suppose we wanted to find $p_0, p_1, p_2, p_3, \dots, p_{10}$. We look at:

```
Expand[(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^10) (1 + x^2 + x^4 + x^6 + x^8 + x^10) (1 + x^3 + x^6 + x^9)
  (1 + x^4 + x^8) (1 + x^5 + x^10) (1 + x^6) (1 + x^7) (1 + x^8) (1 + x^9) (1 + x^10)]
```

$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 54x^{11} + 70x^{12} + 91x^{13} + 116x^{14} + 145x^{15} + 181x^{16} + 222x^{17} + 270x^{18} + 325x^{19} + 386x^{20} + 454x^{21} + 529x^{22} + 616x^{23} + 707x^{24} + 805x^{25} + 910x^{26} + 1022x^{27} + 1135x^{28} + 1255x^{29} + 1374x^{30} + 1497x^{31} + 1618x^{32} + 1741x^{33} + 1856x^{34} + 1966x^{35} + 2069x^{36} + 2165x^{37} + 2246x^{38} + 2319x^{39} + 2379x^{40} + 2425x^{41} + 2456x^{42} + 2473x^{43} + 2473x^{44} + 2456x^{45} + 2425x^{46} + 2379x^{47} + 2319x^{48} + 2246x^{49} + 2165x^{50} + 2069x^{51} + 1966x^{52} + 1856x^{53} + 1741x^{54} + 1618x^{55} + 1497x^{56} + 1374x^{57} + 1255x^{58} + 1135x^{59} + 1022x^{60} + 910x^{61} + 805x^{62} + 707x^{63} + 616x^{64} + 529x^{65} + 454x^{66} + 386x^{67} + 325x^{68} + 270x^{69} + 222x^{70} + 181x^{71} + 145x^{72} + 116x^{73} + 91x^{74} + 70x^{75} + 54x^{76} + 42x^{77} + 30x^{78} + 22x^{79} + 15x^{80} + 11x^{81} + 7x^{82} + 5x^{83} + 3x^{84} + 2x^{85} + x^{86} + x^{87}$

Here we see $p_0 = 1$ $p_1 = 1$ $p_2 = 2$ $p_3 = 3$ $p_4 = 5$ $p_5 = 7$ $p_6 = 11$, $p_7 = 15$ $p_8 = 22$, $p_9 = 30$ and $p_{10} = 42$. However the coefficient of x^{11} does not give p_{11} because the x^{11} term is missing from the first factor. Also note the $1 \cdot x^{87}$ term. This product counts only one partition of 87, namely the one corresponding to the highest powers in the factors, i.e. 10, 10, 9, 8, 10, 6, 7, 8, 9, 10, i.e. 10 1's, 5 2's, 3 3's, 2 4's, 2 5's, 1 6, 1 7, 1 8 & 1 9.

Example Find generating function for h_n = (# of partitions of n into distinct parts)

Ex $6 = 1+5$ $6 = 1+5$ $6 = 2+4$ $6 = 6$ $6 = 1+2+3$

$$g(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

$$= \frac{1-x^2}{1-x}\,\frac{1-(x^2)^2}{1-x^2}\,\frac{1-(x^3)^2}{1-x^3}\,\frac{1-(x^4)^2}{1-x^4}\,\frac{1-(x^5)^2}{1-x^5}\cdots$$

$$= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\,\frac{1-x^{10}}{1-x^5}\cdots$$

$$= \frac{1}{1-x}\,\frac{1}{1-x^3}\,\frac{1}{1-x^5}\cdots = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$$

Read the material on orderings of partitions, although we will not need it.

For instance the lexicographic ordering of partitions of n is the usual "dictionary" ordering

<u>Ex</u>

$$\left.\begin{array}{ll} & 3\,3\,3\,2\,1\,1\,1\,1 \\ \le & 3\,3\,3\,3\,2\,1\,0\,0 \end{array}\right\} \text{partitions of } 15$$

Ordering of partitions of 5

1 1 1 1 1
2 1 1 1
2 2 1
3 1 1
3 2
4 1
5