§ 8.2 Sterling Numbers Continued

Recall:

$$n^p = \sum_{k=0}^{p} S(p,k)\, P(n,k)$$

and $S(p,k) = \left(\begin{array}{l}\text{\# of partitions of } \{1,2,3,\dots p\} \text{ into} \\ k \text{ indistingushable boxes, with} \\ \text{no box empty}\end{array}\right)$

Also $S(p,k)$ is called a Sterling number of the second kind

Today's goals

1. Develop a formula for $S(p,k)$
2. Introduce $s(p,k)$, Sterling numbers of the first kind.
3. Discover what $s(p,k)$ counts.

Goal 1

First let $S^{\#}(p,k) = \left(\begin{array}{l}\text{\# of partitions of } \{1,2,3,\dots p\} \text{ into} \\ \text{boxes } B_1\, B_2 \cdots B_k \text{ with no box empty}\end{array}\right)$

Thus $S^{\#}(p,k) = k!\, S(p,g)$ so $S(p,k) = \frac{1}{k!} S^{\#}(p,k)$

Strategy: Develop formula for $S^{\#}(p,k)$ to get one for $S(p,k)$.

Let $\mathcal{U} = \left\{ (B_1, B_2, \dots, B_k) \,\middle|\, B_i \subseteq \{1,2,3\dots p\},\ B_i \cap B_j = \phi,\ \bigcup_{i=1}^{k} B_i = \{1,2,\dots p\} \right\}$

= Set of partitions of $\{1,2,3,\dots p\}$ into boxes $B_1\, B_2 \dots B_k$.

Then $|\mathcal{U}| = k^p = \left(\text{\# of functions } \{1,2,3,\dots p\} \to \{B_1\, B_2 \cdots B_k\}\right)$

Put $A_i = \{(B_1 B_2 \dots B_k) \in \mathcal{U} \mid B_i = \phi\}$.

Seek

$$\begin{aligned} S^{\#}(p,k) &= |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k| \\ &= |\overline{A_1 \cup A_2 \cup \cdots \cup A_k}| \\ &= |\mathcal{U}| - |A_1 \cup A_2 \cup \cdots \cup A_k| \end{aligned}$$

$$= k^p - \left(\sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_l| - \cdots \right)$$

$$= k^p - \binom{k}{1}(k-1)^p + \binom{k}{2}(k-2)^p - \binom{k}{3}(k-3)^p + \cdots$$

$$= \binom{k}{0}(k-0)^p - \binom{k}{1}(k-1)^p + \binom{k}{2}(k-2)^p - \binom{k}{3}(k-3)^p + \cdots$$

$$= \sum_{t=0}^{p} (-1)^t \binom{k}{t}(k-t)^p$$

<u>Theorem 8.2.6</u> $$S(p,k) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (k-t)^p$$

<u>Ex</u> $$S(4,2) = \frac{1}{2!} \sum_{t=0}^{2} (-1)^t \binom{2}{t} (2-t)^4$$

$$= \frac{1}{2}\left(\binom{2}{0} 2^4 - \binom{2}{1} 1^4 + \binom{2}{2} 0^4 \right)$$

$$= \frac{1}{2}(16 - 2 + 0) = \boxed{7} \quad ✓$$

Sterling numbers of the <u>first</u> <u>kind</u>

$\begin{cases} S(p,k) & \text{sterling numbers of the second kind} \\ s(p,k) & \text{sterling numbers of the first kind} \end{cases}$

<u>Basic Idea:</u>

$$n^p = \sum_{k=0}^{p} S(p,k)\, P(n,k)$$

$$P(n,p) = \sum_{k=0}^{p} (-1)^{p-k} s(p,k)\, n^k$$

$$P(n,p+1) = P(n,p)(n-p)$$

$$P(n,0) = 1$$

$$P(n,1) = n$$

$$P(n,2) = n(n-1) = n^2 - n + 0$$

$$P(n,3) = n(n-1)(n-2) = n^3 - 3n^2 + 2n - 0$$

$$S(p,0) = 0$$

$$S(p,p) = 1 \quad \text{for } p>0$$

$$P(n,4) = (n^3 - 3n^2 + 2n)(n-3)$$

$$= n^4 - 3n^3 + 2n^2$$

$$-3n^3 + 9n^2 - 6n \quad = n^4 - 6n^3 + 11n^2 - 6n + 0$$

$$= 0n^0 - 6n + 11n^2 - 6n^3 + 1n^4$$

$S(4,0)=0$ $\quad S(4,1)=6$ $\quad S(4,2)=11$ $\quad S(4,3)=6$ $\quad S(4,4)=1$

We can get the following recurrence for $S(p,k)$.

$$P(n,p+1) = P(n,p)(n-p)$$

$$= \left(\sum_{k=0}^{p} (-1)^{p-k} S(p,k)\, n^k \right)(n-p)$$

$$\sum_{k=0}^{p} (-1)^{p-k} S(p,k)\, n^{k+1} - \sum_{k=0}^{p} (-1)^{p-k} p\, S(p,k)\, n^k$$

$$\sum_{k=1}^{p+1} (-1)^{p+1-k} S(p,k-1)\, n^k + \sum_{k=0}^{p} (-1)^{p+1-k} p\, S(p,k)\, n^k$$

$$= \sum_{k=0}^{p+1} (-1)^{p+1-k} \Big(\underbrace{S(p,k-1) + p\, S(p,k)}_{S(p+1,\,k)} \Big) n^k$$

$$\Rightarrow \quad S(p+1,k) = S(p,k-1) + p\, S(p,k) \quad \text{for } 0 < k < p+1$$

Theorem 8.2.8 $s(p,0)=0$ and $s(p,p)=1$ for $p>0$. Otherwise:

$$s(p+1,k) = s(p,k-1) + p\,s(p,k)$$

or: $$s(p,k) = s(p-1,k-1) + (p-1)\,s(p-1,k)$$

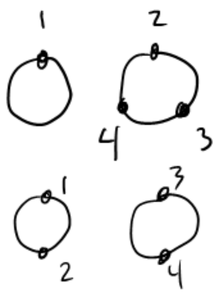
Table for $s(p,k)$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$p=0$	1					
$p=1$	0	1				
$p=2$	0	1	1			
$p=3$	0	2	3	1		
$p=4$	0	6	11	6	1	
$p=5$	0	24	50	35	10	1

Theorem 8.2.9

$$s(p,k) = \begin{pmatrix}\text{\# of arrangements of } p \text{ things} \\ \text{into } k \text{ circular permutations}\end{pmatrix}$$

Example $s(4,2)=11$



$\longleftarrow$ $4 \cdot 2 = 8$ of these

$\longleftarrow$ 3 of these

Proof Let $s^*(p,k) = \begin{pmatrix}\text{\# of arrangements of } p \text{ things} \\ \text{into } k \text{ circular permutations}\end{pmatrix}$

Note $s^*(p,0) = 0 = s(p,0)$ for $p>0$

$s^*(p,p) = 1 = s(p,p)$ for $p \geq 0$.

Also $s^*(p+1,k) = s(p,k-1) + p\,s(p,k)$

$s(p,k)$ arrangements of $\{1,2,3,\dots p\}$ into k circular perms.

Can add $p+1$ to left of any of the p existing elements

$p+1$

$k-1$ circular perms of p things