§ 7.4 Generating Functions (Continued)

Definition: Let $h_0\ h_1\ h_2\ h_3 \cdots$ be an infinite sequence of numbers. (Usually h_n is the answer of some counting question involving n, e.g. how many non-neg. integer solutions to $x+y+z=n$?) The generating function for this sequence is

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots$$

Generating functions often take the forms

Newton's Binomial Formula

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$$

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots + \binom{\alpha}{k}x^k + \cdots$$

$$(1-x)^\alpha = 1 - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \binom{\alpha}{3}x^3 + \cdots + (-1)^k\binom{\alpha}{k}x^k + \cdots$$

If $\alpha = -n < 0$, $\binom{-n}{k} = \dfrac{(-n)(-n-1)(-n-2)(-n-3)\cdots(-n-k+1)}{k!}$

$$= (-1)^k \frac{(n+k-1)(n+k)(n+k-1)\cdots(n)}{k!} = (-1)^k\binom{n+k-1}{k}$$

$$(1-x)^{-n} =$$

$$\frac{1}{(1-x)^n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots + \binom{n+k-1}{k}x^k + \cdots$$

$$\frac{1}{(1-rx)^n} = 1 + \binom{n}{1}rx + \binom{n+1}{2}r^2x^2 + \binom{n+2}{3}r^3x^3 + \cdots + \binom{n+k-1}{k}r^kx^k + \cdots$$

Also necessary:

$$(a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots)(b_0 + b_1x^1 + b_2x^2 + b_3x^3 + \cdots) =$$

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots + (a_kb_0 + a_{k-1}b_1 + a_{k-2}b_2 + \cdots + a_0b_k)x^k + \cdots$$

Example How many ways are there to place 25 identical balls in 7 boxes if the first box can contain no more than 10 balls, but the other boxes can contain any amount?

Strategy

- Let $h_n = \begin{pmatrix} \text{\# of ways to put } n \text{ balls in 7} \\ \text{boxes with } 1^{st} \text{ box containing at most 10} \end{pmatrix}$
- Make generating function $g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots$
- Find h_{25}, the coefficient of x^{25}.

Generating Function

$$g(x) = (1 + x + x^2 + x^3 + \cdots + x^{10})(1 + x + x^2 + x^3 + x^4 + \cdots)^6$$

$$= \frac{1 - x^{11}}{1 - x}\left(\frac{1}{1 - x}\right)^6 = \frac{1}{(1-x)^7}(1 - x^{11})$$

$$= \left(1 + \binom{7}{1}x + \binom{8}{2}x^2 + \binom{9}{3}x^3 + \cdots + \binom{7+k-1}{k}x^k + \cdots\right)\left(1 + 0x + 0x^2 + \cdots - 1 \cdot x^{11} + 0x^{12} + \cdots\right)$$

The x^{25} term of this product is

$$\left(\binom{7+n-1}{n} \cdot 1 + \cdots - \binom{7+n-11-1}{n-11}(-1)\right)x^n = \left(\binom{6+n}{n} - \binom{n-5}{n-11}\right)x^n$$

Thus the number of ways to put n balls in 7 boxes with 1^{st} box containing no more than 10 balls is:

$$\binom{6+n}{n} - \binom{n-5}{n-11}$$

When $n = 25$ we get the answer to our question

$$\underline{\text{Ans}} \quad \binom{6+25}{25} - \binom{n-5}{n-11} = \binom{31}{25} - \binom{20}{14} =$$

$$\frac{31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26}{6!} - \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{6!} =$$

$$\frac{31 \cdot \cancel{30} \cdot 29 \cdot 7 \cdot 9 \cdot 13}{(6 \cdot 5) \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} - \frac{\cancel{20} \cdot 19 \cdot \cancel{18} \cdot 17 \cdot 8 \cdot 15}{\cancel{6} \cdot (\cancel{5 \cdot 4}) \cdot \cancel{3} \cdot \cancel{2}} =$$

$$736281 - 87210 = \boxed{697521}$$

§7.5 Recurrences and Generating Functions.

Goal Use generating functions to solve recurrence relations.

Example Consider sequence 1, 1, 5, 13, 41, 121, ... given by $h_n = 2h_{n-1} + 3h_{n-2}$.

Here is a technique that solves this recurrence relations.

Generating function for this sequence is

$$g(x) = 1 + 1x + 5x^2 + 13x^3 + 41x^4 + 121x^5 + \cdots$$
$$= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots$$

$$\begin{aligned}
g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + h_5 x^5 + \cdots \\
-2x\,g(x) &= \quad -2h_0 x - 2h_1 x^2 - 2h_2 x^3 - 2h_3 x^4 - 2h_4 x^5 - \cdots \\
-3x^2 g(x) &= \quad\quad -3h_0 x^2 - 3h_1 x^3 - 3h_2 x^4 - 3h_3 x^5 - \cdots
\end{aligned}$$

$$g(x) - 2x\,g(x) - 3x^2 g(x) = h_0 + h_1 x - 2h_0 x$$

$$g(x)(1 - 2x - 3x^2) = 1 + x - 2x$$

$$g(x) = \frac{1 - x}{1 - 2x - 3x^2} = \frac{1 - x}{(1 + x)(1 - 3x)}$$

This is the generating function for the sequence. The numbers $h_0\ h_1\ h_2 \cdots$ are the coefficients of its Taylor series. We will now find a formula for these numbers, i.e. a solution to the reccurence relation.

First we will break $g(x)$ into partial fractions so we can get control over those simpler parts.

$$g(x) = \frac{1-x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}$$

$$1 - x = A(1+x) + B(1-3x)$$
$$1 - x = (A+B) + (A-3B)x$$

$$\begin{cases} A + B = 1 \\ A - 3B = -1 \end{cases} \Rightarrow -4B = -2 \Rightarrow B = \frac{1}{2}$$
$$\Rightarrow A = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore

$$g(x) = \frac{1}{2}\,\frac{1}{1+x} + \frac{1}{2}\,\frac{1}{1-3x}$$

$$= \frac{1}{2}\,\frac{1}{1-(-x)} + \frac{1}{2}\,\frac{1}{1-3x}$$

$$= \frac{1}{2}\left(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\right) + \frac{1}{2}\left(1 + 3x + 3^2x^2 + 3^3x^3 + \cdots\right)$$

Coefficient of x^n is $\frac{1}{2}(-1)^n + \frac{1}{2}3^n$

Because $g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots$ and we just found the coefficient of x^n, we get our solution

$$\boxed{h_n = \frac{1}{2}(-1)^n + \frac{1}{2}3^n}$$