exists an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, K_{n_3}.$$

In words, if each of the edges of K_p is colored red, blue, or green, then either there is a red K_{n_1} or a blue K_{n_2} or a green K_{n_3}. The smallest integer p for which this assertion holds is the Ramsey number $r(n_1, n_2, n_3)$. The only non-trivial Ramsey number of this type that is known is

$$r(3,3,3) = 17.$$

The Ramsey numbers $r(n_1, n_2, \ldots, n_k)$ are defined in a similar way, and Ramsey's theorem in its full generality for pairs asserts that these numbers exist; that is, there is an integer p such that

$$K_p \to K_{n_1}, K_{n_2}, \ldots, K_{n_k}.$$

There is a more general form of Ramsey's theorem in which pairs (subsets of two elements) are replaced by subsets of t elements for some fixed integer $t \geq 1$. Let

$$K_n^t$$

denote the collection of all subsets of t elements of a set of n elements. Generalizing our notation above, the general form of Ramsey's theorem asserts: Given integers $t \geq 2$ and integers $q_1, q_2, \ldots, q_k \geq t$, there exists an integer p such that

$$K_p^t \to K_{q_1}^t, K_{q_2}^t, \ldots, K_{q_k}^t.$$

In words, there exists an integer p such that if the each of the t-element subsets of a p element set is assigned one of k colors $c_1, c_2, \ldots,$ and c_k, then either there are q_1 elements all of whose t element subsets are assigned the color c_1, or there are q_2 elements all of whose t-element subsets are assigned the color c_2, . . . , or there are q_k elements all of whose t element subsets are assigned the color c_k. The smallest such integer p is the *Ramsey number*

$$r_t(q_1, q_2, \ldots, q_k).$$

Suppose $t = 1$. Then $r_1(q_1, q_2, \ldots, q_k)$ is the smallest number p such that if the elements of a set of p elements are colored with one of the colors $c_1, c_2, \ldots, c_k$, then either there are q_1 elements of color c_1, or q_2 elements of color c_2, or . . . , or q_k elements of color c_k. Thus, by the strong form of the pigeonhole principle,

$$r_1(q_1, q_2, \ldots, q_k) = q_1 + q_2 + \cdots q_k - k + 1.$$

This demonstrates that Ramsey's theorem is a generalization of the strong form of the pigeonhole principle.

The determination of the general Ramsey numbers $r_t(q_1, q_2, \ldots, q_k)$ is a difficult problem. Very little is known about their exact values. It is not difficult to see that

$$r_t(t, q_2, \ldots, q_k) = r_t(q_2, \ldots, q_k),$$

and that the order in which $q_1, q_2, \ldots, q_k$ are listed does not affect the value of the ramsey number.

2.4 Exercises

1. Concerning Application 4, show that there is a succession of days during which the chess master will have played exactly k games, for each $k = 1, 2, \ldots, 21$. (The case $k = 21$ is the case treated in Application 4.) Is it possible to conclude that there is a succession of days during which the chess master will have played exactly 22 games?

2. $*$ Concerning Application 5, show that if 100 integers are chosen from $1, 2, \ldots, 200$, and one of the integers chosen is less than 16, then there are two chosen numbers such that one of them is divisible by the other.

3. Generalize Application 5 by choosing (how many?) integers from the set
$$\{1, 2, \ldots, 2n\}.$$

4. Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by 1.

5. Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.

6. Generalize Exercises 4 and 5.

7. ∗ Show that for any given 52 integers there exist two of them whose sum, or else whose difference, is divisible by 100.

8. Use the pigeonhole principle to prove that the decimal expansion of a rational number m/n eventually is repeating. For example,

$$34,478/99,900 = .34512512512512512\cdots.$$

9. In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?

10. A child watches TV at least one hour each day for 7 weeks but never more than 11 hours in any one week. Prove that there is some period of consecutive days in which the child watches exactly 20 hours of TV. (It is assumed that the child watches TV for a whole number of hours each day.)

11. A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.

12. Show by example that the conclusion of the Chinese remainder theorem (Application 6) need not hold when m and n are not relatively prime.

13. ∗ Let S be a set of 6 points in the plane, with no 3 of the points collinear. Color either red or blue each of the 15 line segments determined by the points of S. Show that there are at least two triangles determined by points of S which are either red triangles or blue triangles. (Both may be red, or both may be blue, or one may be red and the other blue.)

14. A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. If I pick one piece of fruit out of the bag every minute, how long will it be before I am assured of having picked at least a dozen pieces of fruit of the same kind?

15. Prove that for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

16. Prove that in a group of $n > 1$ people there are two who have the same number of acquaintances in the group. (It is assumed that no one is acquainted with him or herself.)

17. There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

18. Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.

19. (a) Prove that of any five points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{2}$.

    (b) Prove that of any ten points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.

    (c) Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/n$.

20. Prove that $r(3,3,3) \leq 17$.

21. ∗ Prove that $r(3,3,3) \geq 17$ by exhibiting a coloring, with colors red, blue, and green, of the line segments joining 16 points with the property that there do not exist 3 points such that the 3 line segments joining them are all colored the same.

22. Prove that

$$r(\underbrace{3,3,\ldots,3}_{k+1}) \leq (k+1)(r(\underbrace{3,3,\ldots,3}_{k}) - 1) + 2.$$

Use this result to obtain an upper bound for

$$r(\underbrace{3,3,\ldots,3}_{n}).$$

23. The line segments joining 10 points are arbitrarily colored red or blue. Prove that there must exist 3 points such that the 3 line segments joining them are all red, or 4 points such that the 6 line segments joining them are all blue (that is, $r(3,4) \leq 10$).

24. Let q_3 and t be positive integers with $q_3 \geq t$. Determine the Ramsey number $r_t(t, t, q_3)$.

25. Let $q_1, q_2, \ldots, q_k, t$ be positive integers where $q_1 \geq t$, $q_2 \geq t, \ldots,$ $q_k \geq t$. Let m be the largest of $q_1, q_2, \ldots, q_k$. Show that

$$r_t(m, m, \ldots, m) \geq r_t(q_1, q_2, \ldots, q_k).$$

Conclude that to prove Ramsey's theorem it is enough to prove it in the case that $q_1 = q_2 = \cdots = q_k$.

26. Suppose that the mn people of a marching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to her or his left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.

27. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

Chapter 3

Permutations and Combinations

Most readers of this book will have had some experience with simple counting problems, so that the concepts "permutations" and "combinations" are probably familiar. But the experienced counter knows that even rather simple-looking problems can pose difficulties in their solutions. While it is generally true that in order to learn mathematics one must *do* mathematics, it is especially so here—the serious student should attempt to solve a large number of problems.

In this chapter we explore two general principles and some of the counting formulas that they imply.

3.1 Two Basic Counting Principles

The first principle is elementary. It is one formulation of the principle that the whole is equal to the sum of its parts.

A *partition* of a set S is a collection $S_1, S_2, \ldots, S_m$ of subsets of S such that each element of S is in exactly one of those subsets:

$$S = S_1 \cup S_2 \cup \cdots \cup S_m,$$

$$S_i \cap S_j = \emptyset, \quad (i \neq j).$$

The subsets $S_1, S_2, \ldots, S_m$ are called the *parts* of the partition. We note that a part of a partition may be empty, but usually there is no advantage in considering partitions with one or more empty parts. The number of objects of a set S is denoted by $|S|$ and is sometimes called the *size* of S.