VCU

MATH 504

Algebraic Structures and Functions

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Test 1

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Score: 100

Directions. Answer the questions in the space provided. Unless noted otherwise, you must show and explain your work to receive full credit. In proofs, justify each step to the extent reasonable.

This is a closed-book, closed-notes test. Calculators, computers, etc., are not used.

There are 10 numbered questions; each is worth 10 points.

1. The following is a partial table for a **commutative** and **associative** binary operation on a set $S = \{a, b, c, d\}$.

$*$	a	b	c	d
a	d	c		
b	c	d	a	
c		a		
d				

because operation is commutative we can extend table like this

Supply the following information. Briefly justify each answer.

(a) $a * b = \boxed{c}$ (from table)

(b) $c * b = \boxed{a}$ (from table)

(c) $d * a = (b * b) * a = b * (b * a) = b * c = \boxed{a}$

(d) $d * c = (b * b) * c = b * (b * c) = b * a = \boxed{c}$

2. State the definition of a group.

A group is a set G on which there is defined a binary operation $*$ that satisfies each of the following conditions:

$\mathcal{G}_1$: The operation $*$ is associative, i.e. $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$.

$\mathcal{G}_2$: There is an element $e \in G$ for which $e * x = x = x * e$ for all $x \in G$.

$\mathcal{G}_3$: For each $x \in G$ there is also an element $x' \in G$ satisfying $x' * x = e = x * x'$.

3. For each of the following binary structures, say whether or not it is a group. If a structure is not a group, name at least one group axiom $\mathscr{G}_1$, $\mathscr{G}_2$, or $\mathscr{G}_3$ that fails.

(a) $\langle \mathbb{Z}_5, +_5 \rangle$ Group

(b) $\langle \mathbb{Q}^*, \div \rangle$ NOT group. Not associative

$4 = (8 \div 1) \div 2 \neq 8 \div (1 \div 2) = 16$

(c) $\langle \mathbb{Q}, + \rangle$ Group

(d) $\langle \mathbb{Q}^*, \cdot \rangle$ Group

(e) $\langle \mathbb{Z}^*, \cdot \rangle$ NOT group. The inverse of 2 would be $\frac{1}{2}$ but $\frac{1}{2} \notin \mathbb{Z}^*$.

(f) $\langle 3\mathbb{Z}, + \rangle$ Group

(g) The unit circle $U = \{z \mid z \in \mathbb{C} \text{ and } |z| = 1\}$ with the binary operation of multiplication of complex numbers.

Group

(h) The set of all bijective functions $f : \mathbb{R} \to \mathbb{R}$ with the binary operation of function composition.

Group

(i) $M_{2\times 2}(\mathbb{R})$ with matrix multiplication.

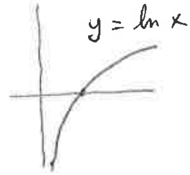
NOT group $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse.

(j) $M_{2\times 2}(\mathbb{R})$ with matrix addition.

Group

4. Are the groups $\langle \mathbb{R}^+, \cdot \rangle$ and $\langle \mathbb{R}, + \rangle$ isomorphic? Explain.

Consider the function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined as $\varphi(x) = \ln(x)$.
We know from calculus that φ is a bijection.



Also $\varphi(x \cdot y) = \ln(xy)$
$= \ln(x) + \ln(y) = \varphi(x) + \varphi(y)$

Consequently φ is an isomorphism

5. Suppose $\varphi : \mathbb{Z}_8 \to U_8$ is an isomorphism satisfying $\varphi(3) = \dfrac{1-i}{\sqrt{2}}$. Find $\varphi(6)$.

$$\varphi(6) = \varphi(3+3) = \varphi(3) \cdot \varphi(3)$$
$$= \frac{1-i}{\sqrt{2}} \; \frac{1-i}{\sqrt{2}}$$
$$= \frac{1-2i-1}{2}$$
$$= \boxed{-i}$$

6. Write the multiplication tables for $\mathbb{Z}_4$ and the Klein 4-group.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

7. Suppose G is a group that has no subgroups other than $H = \{e\}$ and $H = G$. Prove that G must be cyclic.

Proof Suppose that the only subgroups of G are $H = \{e\}$ and $H = G$. Now take any $a \in G$ with $a \neq e$. Form the cyclic subgroup $H = \langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$. Now, this subgroup contains both a and e so $H \neq \{e\}$. But The only other subgroup of G is G, so it must be that $H = \langle a \rangle = G$.

But this means $G = \langle a \rangle$, so G is cyclic, generated by a ▨

8. Suppose G is a group having the property that $(ab)^{-1} = a^{-1}b^{-1}$ for each $a, b \in G$. Prove that G is abelian.

Proof (Direct)

Suppose G has the property that $(ab)^{-1} = a^{-1}b^{-1}$ for each $a, b \in G$

Take any two $a, b \in G$. We have

$$(ab)^{-1} = a^{-1}b^{-1}. \qquad (*)$$

Now $(ab)^{-1} = b^{-1}a^{-1}$, by the formula for the inverse of a product. Combining this with $(*)$ gives

$$b^{-1}a^{-1} = a^{-1}b^{-1}.$$

Thus $(b^{-1}a^{-1})^{-1} = (a^{-1}b^{-1})^{-1}$

So $(a^{-1})^{-1}(b^{-1})^{-1} = (b^{-1})^{-1}(a^{-1})^{-1}$

i.e. $ab = ba$.

Now we've shown $ab = ba$ for any $a, b \in G$, so G is abelian. ▨

9. Suppose K is a subgroup of an abelian group G.
Prove that the set $H = \{x \in G \mid x^2 \in K\}$ is a subgroup of G.

Proof First note that the set H is closed. Suppose $x, y \in H$. This means $x^2 \in K$ and $y^2 \in K$. Now consider xy and note that $(xy)^2 = xyxy$ and so $(xy)^2 = xxyy = x^2y^2$ by the abelian property. Now since $(xy)^2 = x^2y^2$ and both x^2 and y^2 are in K and K is closed (it is a subgroup) it follows that $(xy)^2 = x^2y^2 \in K$. Therefore $xy \in H$ by definition of H, so we have shown H is closed.

Next note that $e^2 = e \in K$, which means $e \in H$.

Finally suppose $x \in H$. We need to show $x^{-1} \in H$. Because $x \in H$, we know $x^2 \in K$, or $xx \in K$. But since K is a subgroup it must the be also true that $(xx)^{-1} \in K$. By the inverse of a product this becomes $x^{-1}x^{-1} \in K$, or $(x^{-1})^2 \in K$. Thus $x^{-1} \in H$.

The above reasoning shows that H is indeed a subgroup of G.

10. Draw the subgroup lattice for $\mathbb{Z}_{12}$.
(Just draw the diagram; you do not need to justify or show any work.)