On the first homework assignment most of the class struggled with #18. Many did not attempt it. We can expect lots of proof-type problems like this one, so we’ll devote some class time to its solution.

In this problem \( A \) is an arbitrary set, \( B = \{0,1,3\} \) and \( B^A \) is the set of all functions \( f: A \to B \). We are asked to show \( |B^A| = |P(A)| \). To accomplish this we must find a one-to-one and onto function \( \Phi: B^A \to P(A) \).

To get an idea of how to proceed, look at the example where \( A = \{a, b\} \). Let’s list out \( B^A \) and \( P(A) \).

\[
B^A = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}, \quad P(A) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}
\]

Notice that indeed \( |B^A| = 4 = |P(A)| \). But also the example suggests an actual function \( \Phi: B^A \to P(A) \) sending any function \( f \in B^A \) to the set \( \{ x \in A \mid f(x) = 1 \} \) in \( P(A) \), (just below \( f \)).

Now that we’ve got an idea, let’s put it all together.

**So #18** Suppose \( A \) is any set (finite or infinite) and \( B^A \) is the set of all functions \( f: A \to B \). Prove that \( |B^A| = |P(A)| \).

**Proof** We need to produce a one-to-one and onto function \( \Phi: B^A \to P(A) \). To do this, let \( \Phi: B^A \to P(A) \) be the function defined as \( \Phi(f) = \{ x \in A \mid f(x) = 1 \} \).

First note that \( \Phi \) is one-to-one: Suppose \( f, g \in B^A \) and \( f \neq g \).

We want to show \( \Phi(f) \neq \Phi(g) \). Now, \( f \neq g \) means that there is an element \( a \in A \) for which \( f(a) \neq g(a) \). Then either \( f(a) = 0 \) and \( g(a) = 1 \), or \( f(a) = 1 \) and \( g(a) = 0 \). Let’s say \( f(a) = 0 \) and \( g(a) = 1 \). (The other case is nearly identical.) Now since \( f(a) = 0 \), we know \( a \notin \{ x \in A \mid f(x) = 1 \} = \Phi(f) \).

And because \( g(a) = 1 \), we know \( a \in \{ x \in A \mid g(x) = 1 \} = \Phi(g) \). Therefore \( a \notin \Phi(f) \) but \( a \in \Phi(g) \), so \( \Phi(f) \neq \Phi(g) \).

Next note that \( \Phi \) is onto: Take an arbitrary set \( X \in P(A) \), which is to say \( X \) is an arbitrary subset \( X \subseteq A \). Now construct a function \( f: A \to B \) defined as \( f(x) = \begin{cases} 0 & \text{if } x \notin X \\ 1 & \text{if } x \in X \end{cases} \).

Then \( f \in B^A \) and note that \( f \) has been constructed so that \( \Phi(f) = \{ x \in A \mid f(x) = 1 \} = X \). Therefore \( \Phi \) is onto.