PART II  Permutations  Cosets, Direct Products
Section 8  Groups of Permutations

Today we examine how permutations have a group structure and how every group can be viewed in terms of permutations.

Intuitive idea:
A permutation of objects is a rearrangement of them on a line
\[
\begin{array}{ccccccc}
    a & b & c & d & e & f \\
    \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
    e & b & f & a & c & d
\end{array}
\]

Definition
A permutation of a set A is a 1-1 and onto function \( \sigma : A \rightarrow A \).

Examples  \( A = \{1, 2, 3\} \)

Permutation of A:

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}
\]

\( \sigma(1) = 2 \quad \sigma(2) = 3 \quad \sigma(3) = 1 \)

Permutation of A:

\[
\tau = \begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\]

\( \tau(1) = 2 \quad \tau(2) = 1 \quad \tau(3) = 3 \)

Consider  \( \sigma \circ \tau \)

\[
\begin{array}{ccc}
\sigma \circ \tau(1) &=& \sigma(\tau(1)) = \sigma(2) = 3 \\
\sigma \circ \tau(2) &=& \sigma(\tau(2)) = \sigma(3) = 2 \\
\sigma \circ \tau(3) &=& \sigma(\tau(3)) = \sigma(1) = 1
\end{array}
\]

\[
\sigma \circ \tau = \begin{pmatrix}
1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix}
\]

Consider  \( \tau \circ \sigma \)

\[
\begin{array}{ccc}
\tau \circ \sigma(1) &=& \tau(\sigma(1)) = \tau(2) = 1 \\
\tau \circ \sigma(2) &=& \tau(\sigma(2)) = \tau(3) = 3 \\
\tau \circ \sigma(3) &=& \tau(\sigma(3)) = \tau(1) = 2
\end{array}
\]

\[
\tau \circ \sigma = \begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix}
\]

Note  \( \sigma \circ \tau \) and  \( \tau \circ \sigma \) are permutations of A but  \( \sigma \circ \tau \neq \tau \circ \sigma \).

Notation  \( \sigma \circ \tau = \sigma \tau \)

Notation  \( \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \sigma \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \)

\[
\sigma \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}
\]
Theorem Suppose $A$ is a set. The set of permutations of $A$ is a group under operation $\circ$ (function composition).

1. Function composition is associative.
2. Identity element is $i : A \to A$, $i(x) = x$.
3. Inverse of $\sigma$ is inverse function $\sigma^{-1}$. $\sigma \sigma^{-1}(x) = x$, so $\sigma \sigma^{-1} = i$.

Definition If $A = \{1, 2, 3, \ldots, n\}$, the group of permutations on $A$ is called the symmetric group on $n$ letters and is denoted $S_n$. Thus $|S_n| = n!$.

Example $S_2 = \{ (1 \ 2), (1 \ 2) \}$

Thus $S_2 \cong \mathbb{Z}_2$

Example $S_3 = \{ (1 \ 2 \ 3), (1 \ 2 \ 3), (1 \ 2 \ 3), (1 \ 3 \ 2), (1 \ 3 \ 2), (1 \ 2 \ 3) \}$

Notice that $S_3$ is non-abelian. It turns out to be the smallest non-abelian group.
Cayley's Theorem
Every group is isomorphic to a group of permutations.

The proof uses an idea you've probably observed before:
The rows of a multitable are all permutations of the top row. The idea is to associate each row with its permutation.

\[
\begin{array}{c|ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
\end{array}
\]

Cayley's Theorem
Every group \( G \) is isomorphic to a group of permutations.

Proof (Outline)

Given \( x \in G \), define function \( \lambda_x : G \to G \) as \( \lambda_x(g) = xg \)

Note \( \lambda_x \) is 1-1 \( \lambda_x(a) = \lambda_x(b) \Rightarrow xa = xb \Rightarrow a = b \)

Note \( \lambda_x \) is onto If \( a \in G \) then \( \lambda_x(x^{-1}a) = xx^{-1}a = a \).

Thus \( \lambda_x \) is a permutation of \( G \)

Note \( \lambda_x \lambda_y = \lambda_{xy} \)

Define \( \Phi : G \to (\text{Permutations}) \) as \( \Phi(x) = \lambda_x \)

Then \( \Phi(xy) = \lambda_{xy} = \lambda_x \lambda_y = \Phi(x) \Phi(y) \)

Also check \( \Phi \) is 1-1.
An illustration of Cayley's Theorem

Consider the group $\mathbb{Z}_3$:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

Cayley's Theorem says this is isomorphic to a group of permutations. What group of permutations?

To answer that, note:

\[
\begin{align*}
\pi_0 &= (0 \ 1 \ 2) \\
\pi_1 &= (0 \ 1 \ 2) \\
\pi_2 &= (0 \ 1 \ 2)
\end{align*}
\]

These 3 permutations obey the following multiplication table, which is structurally identical to that of $\mathbb{Z}_3$.

\[
\begin{array}{c|cccc}
\pi & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) \\
\hline
(0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) \\
(0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) \\
(0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) \\
(0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) & (0 \ 1 \ 2) \\
\end{array}
\]