The Fundamental Homomorphism Theorem

Theorem: Suppose \( H \leq G \) is normal. Then \( \gamma : G \rightarrow G/H \) defined as \( \gamma(x) = xH \) is a homomorphism, and \( \ker(\gamma) = H \).

Proof: \( \gamma(xy) = xyH = xHyH = \gamma(x)\gamma(y) \).
\[ \ker(\gamma) = \{ x \in G \mid \gamma(x) = H \} = \{ x \in G \mid xH = H \} = H. \]

Example:

Recall:
Given homomorphism \( \varphi : G \rightarrow K \),
\[ \varphi[G] = \{ \varphi(x) \mid x \in G \} \leq K. \]
\( \varphi \) is onto \( \iff \varphi[G] = K. \)

Theorem: Fundamental Homomorphism Theorem
Suppose \( \varphi : G \rightarrow K \) is a homomorphism with \( H = \ker(\varphi) \).
Then \( \mu : G/H \rightarrow \varphi[G] \) defined as \( \mu(xH) = \varphi(x) \)
is an isomorphism satisfying \( \mu(\gamma(x)) = \varphi(x). \)
Proof

\[ M(xH, yH) = M(xyH) = \phi(xy) = \phi(x) \phi(y) = M(xH)M(yH) \]

\[ \text{ker}(M) = \{ aH \in G/H \mid \mu(aH) = eK \} = \{ a \in G \mid \phi(a) = eK \} = \{ a \in G/H \mid a \in \text{ker}(\phi) \} = \{ eH \} \]

Since \( \text{ker}(M) \) is trivial, \( M \) is \( 1-1 \).

Usually the FHT is applied to situations where \( \phi \) is onto. In such cases it becomes:

**Theorem FHT**

Suppose \( \phi: G \to K \) is an onto homomorphism and \( H = \ker(\phi) \). Then \( \mu: G/H \to G' \) defined as \( \mu(xH) = \phi(x) \) is an isomorphism satisfying \( \mu(\phi(x)) = \phi(x) \).

\[ \begin{array}{ccc} G & \rightarrow & K \\ \phi & \downarrow \mu \quad & \quad \downarrow \phi \\ G/H & \rightarrow & Z_n \end{array} \]