# Section 10 Cosets and Lagranges Theorem

Today we introduce the important notion of cosets. As a point of departure, consider the equivalence relation $\equiv_5$ on $\mathbb{Z}$. Recall this divides the group $\mathbb{Z}$ up into 5 equivalence classes.

$\mathbb{Z}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ... | -10, | -5, | 0, | 5 | 10 | 15 | 20 ... | $\leftarrow 5\mathbb{Z} = H$. |
| ... | -9 | -4 | 1 | 6 | 11 | 16 | 21 ... | $1+H = \{1+h \mid h \in H\}$ |
| | -8 | -3 | 2 | 7 | 12 | 17 | 22 ... | $2+H$ |
| | -7 | -2 | 3 | 8 | 13 | 18 | 23 | $3+H$ |
| | -6 | -1 | 4 | 9 | 14 | 19 | 24 ... | $4+H$ |

These sets are called <u>cosets</u> of subgroup $H = 5\mathbb{Z}$ of $\mathbb{Z}$.

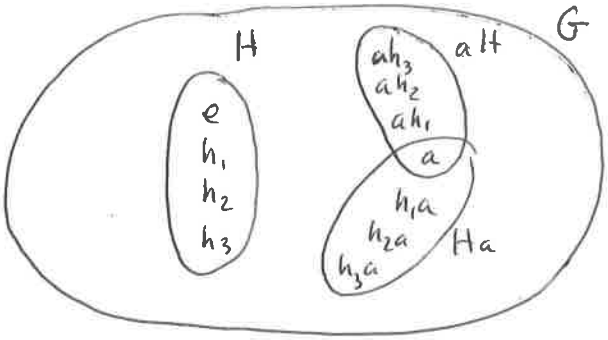
$a+H = \{a+h \mid h \in H\}$. ~~If + not used $aH = \{ah \mid h \in H\}$.~~

$aH = \{ah \mid h \in H\}$ if + not used

<u>Definition</u> If $H \leq G$ and $a \in G$ then

$aH = \{ah \mid h \in H\} \leftarrow$ left coset containing $a$.

$Ha = \{ha \mid h \in H\} \leftarrow$ right coset containing $a$.




If additive notation, $a + H = \{a+h \mid h \in H\} =$
$H + a = \{h+a \mid h \in H\}$

These are equal, so left coset equals right coset.

<u>Warning</u>: If $G$ not abelian, $aH \neq Ha$, in general.

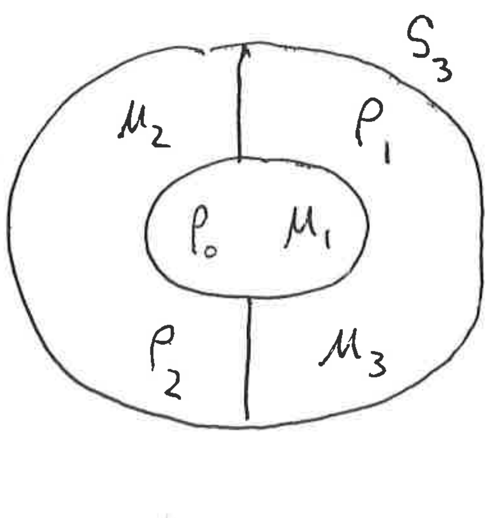
# Example of Left and Right Cosets

Consider $S_3 = \{ \rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3 \}$

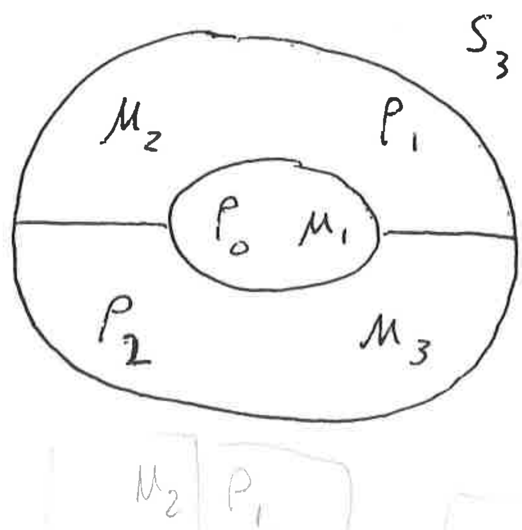
Let $H = \langle \mu_1 \rangle = \{ \rho_0, \mu_1 \}$ be a subgroup of $S_3$.
Following are its left and right cosets.

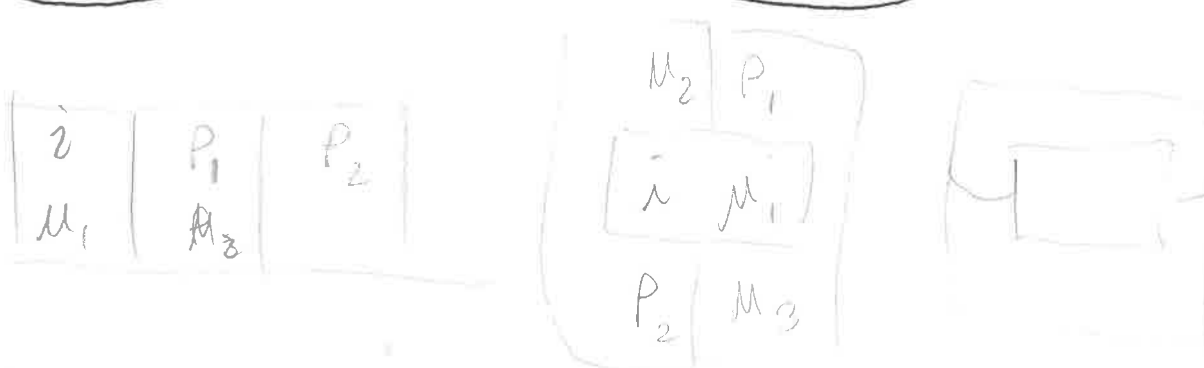
| | |
|---|---|
| $\rho_0 H = \{ \rho_0, \mu_1 \}$ | $H\rho_0 = \{ \rho_0, \mu_1 \}$ |
| $\rho_1 H = \{ \rho_1, \mu_3 \}$ | $H\rho_1 = \{ \rho_1, \mu_2 \}$ |
| $\rho_2 H = \{ \rho_2, \mu_2 \}$ | $H\rho_2 = \{ \rho_2, \mu_3 \}$ |
| $\mu_1 H = \{ \mu_1, \rho_0 \}$ | $H\mu_1 = \{ \mu_1, \rho_0 \}$ |
| $\mu_2 H = \{ \mu_2, \rho_2 \}$ | $H\mu_2 = \{ \mu_2, \rho_1 \}$ |
| $\mu_3 H = \{ \mu_3, \rho_1 \}$ | $H\mu_3 = \{ \mu_3, \rho_2 \}$ |

Left cosets of H




Right cosets of H.



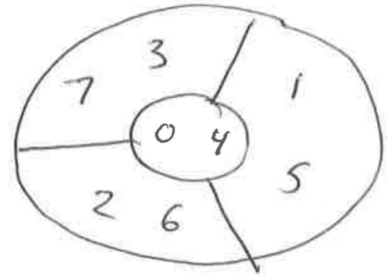
**Ex** Find cosets of $H = \langle 4 \rangle = \{0, 4\}$ in $G = \mathbb{Z}_8$

$0 + H = \{0 + h \mid h \in H\} = \{0, 4\} = 4 + H$

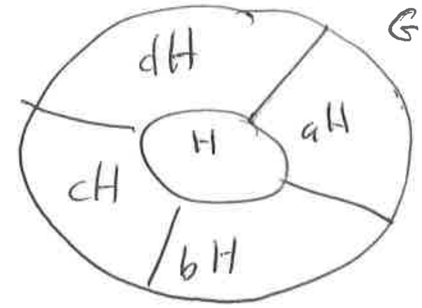
$1 + H = \{1 + h \mid h \in H\} = \{1, 5\} = 5 + H$

$2 + H = \{2 + h \mid h \in H\} = \{2, 6\} = 6 + H$

$3 + H = \{3 + h \mid h \in H\} = \{3, 7\} = 7 + H$




**Key Fact** Right (or left) cosets of $H$ partition $G$ into cells. Any 2 cells have the same number of elements.




**Consequence**

**Lagrange's Theorem** If $G$ is finite and $H \leq G$, then $|H|$ divides $|G|$.

**Ex** Find all subgroups of $\mathbb{Z}_{97}$.

Subgroup $H$ must have $|H|$ divides $97$. $|H| = 1$, or $|H| = 97$.

Subgroup lattice $\{0\}$ — $\mathbb{Z}_{97}$

**Corollary** A group of prime order is cyclic.

**Ex** (Ex 40) If $a \in G$ and $|G| = n$, then $a^n = e$.

**Proof** Let $\langle a \rangle = \{e \ a \ a^2 \ a^3 \ a^4 \cdots a^{m-1}\}$ with $a^m = e$.

Then $|\langle a \rangle| = m$ divides $|G| = n$, i.e. $n = pm$.

$a^n = a^{pm} = (a^m)^p = e^p = e$.

How do you tell if $x, y \in G$ are in the same coset of $H$?

| | | | | |
|---|---|---|---|---|
| ... 0 | 5 | 10 | 15 ... | $H$ |
| ... 1 | 6 | 11 | 15 ... | $1+H$ |
| ... 2 | 7 | 12 | 16 ... | $2+H$ |
| ... 3 | 8 | 13 | 17 ... | $3+H$ |
| ... 4 | 9 | 14 | 18 ... | $4+H$ |

$\mathbb{Z}$

| | | |
|---|---|---|
| 0 | 4 | $H$ |
| 1 | 5 | $1+H$ |
| 2 | 6 | $2+H$ |
| 3 | 7 | $3+H$ |

$\mathbb{Z}_8$

From these examples, we expect $\left(x, y \text{ in same coset}\right) \Longleftrightarrow \left(x - y \in H\right)$

In general $\left(x, y \text{ in same coset}\right) \Longleftrightarrow \left(xy^{-1} \in H\right)$.

But its a little more subtle than this, depending on left & right

**Proposition**

1. $\left(x, y \text{ are in same right coset of } H\right) \Longleftrightarrow xy^{-1} \in H$.
2. $\left(x, y \text{ are in same left coset of } H\right) \Longleftrightarrow x^{-1}y \in H$.

Reason for 1: $\left(x, y \in Ha\right) \Longleftrightarrow \begin{pmatrix} x = ha \\ y = h'a \end{pmatrix} \Longleftrightarrow xy^{-1} = haa^{-1}h'^{-1} = hh'^{-1} \in H$ -

Reason for 2: $\left(x, y \in aH\right) \Longleftrightarrow \begin{pmatrix} x = ah \\ y = ah' \end{pmatrix} \Longleftrightarrow x^{-1}y = h^{-1}a^{-1}ah' = h^{-1}h' \in H$.

This is the way the text does cosets. They define relations as follow

1. $\sim_L$ is a relation on $G$ where $x \sim_L y \Longleftrightarrow x^{-1}y \in H$.
2. $\sim_R$ is a relation on $G$ when $x \sim_R y \Longleftrightarrow xy^{-1} \in H$.

Text: $\sim_L$ is an equiv. relation on $G$. Cells are cosets $aH$ of $H$.

Ex 26 $\sim_R$ is an equiv. relation on $G$. (Cells are cosets $Ha$ of $H$)

$x \in aH$
$x = yh$
$yh \in aH$ $\quad yh = ah'$

$xy^{-1} = h$ $\quad x \in Hy$
$x = hy$ $\quad y = H$

# Index

The index of H in G is $(G:H) = \left(\begin{matrix}\# \text{ of left (or right)} \\ \text{cosets of } H\end{matrix}\right)$

$$(G:H) = \frac{|G|}{|H|} \quad \text{if } |G| < \infty.$$

Ex $(\mathbb{Z} : 5\mathbb{Z}) = 5$

$(\mathbb{Z}_8 : \langle 4 \rangle) = 4$

$(S_3 : A_3) = 2$

**Theorem** If $K \leq H \leq G$, then $(G:K) = (G:H)(H:K)$