Tuesday Section 2, 3
§1 2 4 6 8 10 12 20
Thur Sect 3, 4
§2 2 4 6 10 36

Recall $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

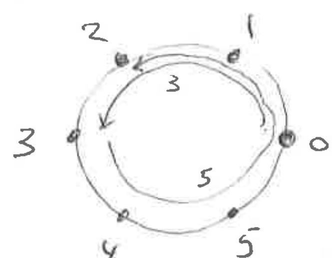
Section 1 Introduction and Examples

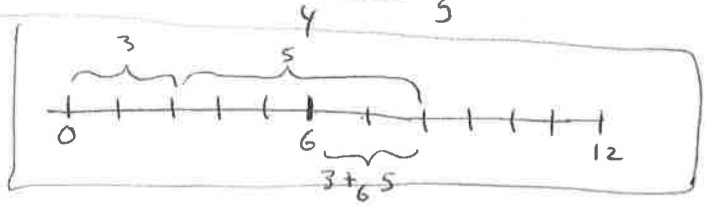
Here we examine a few examples that will be with us for the entire semester and which motivate some of the definitions that come later.

Definition Suppose $n \in \mathbb{Z}^+$.
The *integers modulo n* is the set
$\mathbb{Z}_n = \{0, 1, 2, 3, \cdots, n-1\}$ that are added by the rule $a +_n b$ = (remainder when $a+b$ is divided by n) = $a+b \pmod n$.

Example $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$1 +_6 2 = 3$
$1 +_6 3 = 4$
$1 +_6 4 = 5$
$1 +_6 5 = 0$
$3 +_6 5 = 2$





$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

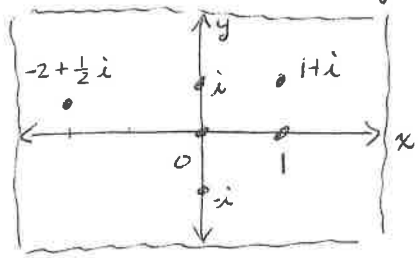
Complex Numbers

Complex numbers are the source of a great many examples in abstract algebra, so let's begin there.

Recall i stands for the special number satisfying $i^2 = -1$.

$\mathbb{C} = \{ x + yi \mid x, y \in \mathbb{R} \}$

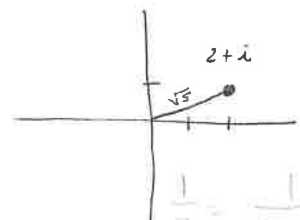
$\mathbb{C}$ can be visualized as the "complex plane"



Addition in $\mathbb{C}$: $(3+4i) + (7-2i) = 10 + 2i$

Multiplication: $(3+4i)(7-2i) = 21 - 6i + 28i - 8i^2 = 29 + 22i$

Notation: $z = x + yi \qquad w = 5 + 3i$



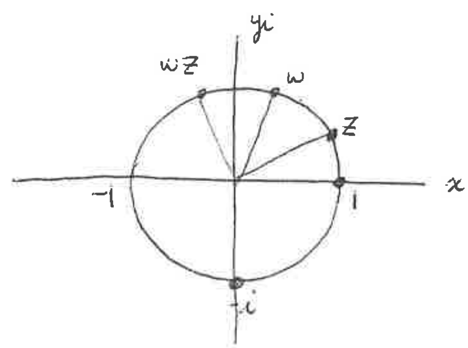
$|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

Absolute value: $|x+iy| = \sqrt{x^2+y^2}$

Exercise: If $z, w \in \mathbb{C}$ then $|zw| = |z||w|$

Algebra on the unit circle

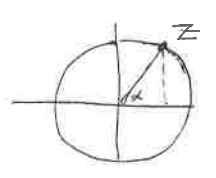
Let $\mathcal{U} = \{ z \in \mathbb{C} \mid |z| = 1 \}$



Notice that if $w, z \in \mathcal{U}$ then
$|wz| = |w||z| = (1)(1) = 1$
so $wz \in \mathcal{U}$.

So the unit circle $\mathcal{U}$ is a little algebraic object unto itself. You can multiply elements of $\mathcal{U}$ and ~~still be in~~ $\mathcal{U}$. the product is still in $\mathcal{U}$. Let's examine this more closely.

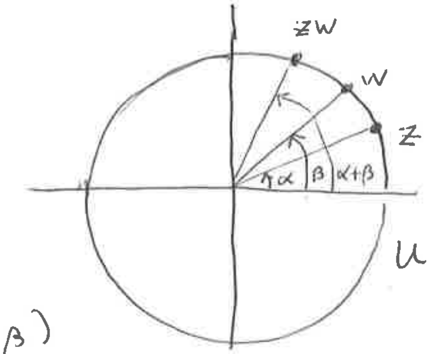
Note If $z \in \mathcal{U}$ then $z = \cos(\alpha) + i\sin(\alpha)$
$= e^{i\alpha}$



"z has polar angle α"

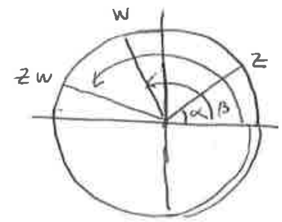
Suppose $z, w \in \mathcal{U} \subseteq \mathbb{C}$

$z = \cos\alpha + i\sin\alpha$
$w = \cos\beta + i\sin\beta$

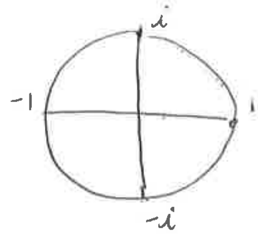


$$zw = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

<u>Conclusion</u> If $z \in \mathcal{U}$ has polar angle α and $w \in \mathcal{U}$ has polar angle β, then $zw \in \mathcal{U}$ has polar angle $\alpha + \beta$

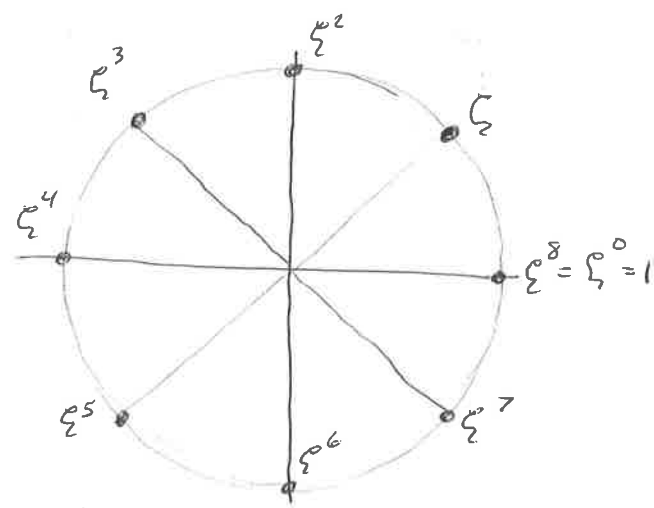


<u>Examples:</u> $(i)(i) = -1$
$(-1)(-i) = -i$
$(-1)(i) = -i$



<u>Roots of Unity</u>

Consider the solutions of $z^8 = 1$.
Notice that one solution is $z = 1$
Notice every solution must have absolute value 1 because if $z^8 = 1$ then $|z|^8 = 1$ ~~because~~
$|zzzzzzzz| = 1 \Rightarrow |z||z||z|\cdots|z| = 1$
$\Rightarrow |z|^8 = 1 \Rightarrow |z| = 1.$



Another solution is $\zeta = \cos\left(\frac{2\pi}{8}\right) + i\sin\left(\frac{2\pi}{8}\right) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

But notice every power of ζ is a solution of $z^8 = 1$:

$$(\zeta^k)^8 = (\zeta^{k8}) = (\zeta^8)^k = (1)^k = 1.$$

Thus the 8 solutions of $z^8 = 1$ are $\mathcal{U}_8 = \{\zeta^0, \zeta^1, \zeta^2, \zeta^3 \cdots \zeta^7\}$

Any two elements of U_8 can be multiplied to get another element of U_8:

$$\zeta^3\zeta^2 = \zeta^5$$

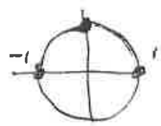
$$\zeta^6\zeta^4 = \zeta^{10} = \zeta^8\zeta^2 = \zeta^2$$

$$\zeta^i\zeta^j = \zeta^{(i+j) \bmod 8}$$

Definitions

The solutions to $x^n = 1$ are called the <u>n^{th} roots of unity</u> $(n \in \mathbb{Z}^+)$.
The set of these solutions is denoted U_n.
If $\zeta = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ then $U_n = \{\zeta^0, \zeta^1, \zeta^2 \cdots \zeta^{n-1}\}$
Elements of U_n are multiplied by rule $\zeta^i\zeta^j = \zeta^{(i+j) \bmod n}$.

Examples $\quad U_2 = \{1, -1\}$



$U_3 = \left\{1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$

$U_4 = \{1, i, -1, -i\}$