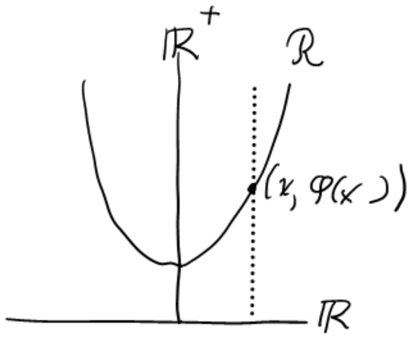
Section 0 (Continued) [Be sure to read about Equivalence Relations]

Functions

Today we will review the notion of a function $f: A \to B$.
In algebra and calculus you dealt with functions $f: \mathbb{R} \to \mathbb{R}$
In advanced mathematics you will encounter functions $f: A \to B$ from a set A to a set B. and this requires a more theoretical view of functions. This involves the idea of a relation from A to B, which we discussed last time.

Definition A relation $\mathcal{R}$ from X to Y is a subset $\mathcal{R} \subseteq X \times Y$.

Now consider the function $\varphi: \mathbb{R} \to \mathbb{R}^+$ defined as $\varphi(x) = x^2+1$. The graph of this function is the set of points $\mathcal{R} = \{(x, \varphi(x)) \mid x \in \mathbb{R}\} = \{(x, x^2+1) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}^+$. This is a relation from the set $\mathbb{R}$ to the set $\mathbb{R}^+$. This suggests the following definition:



Definition
A function $\varphi: X \longrightarrow Y$ is a special kind of relation from X to Y. The requirements are:

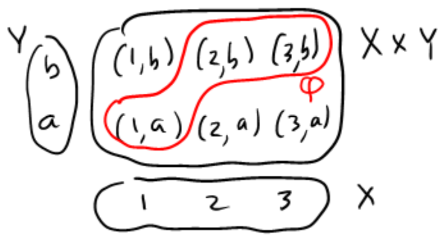
1. $\varphi \subseteq X \times Y$ ⟵ φ is a relation from X to Y
2. Each $x \in X$ occurs in exactly one ordered pair $(x, y) \in \varphi$ ⟵ φ passes the "vertical line test"

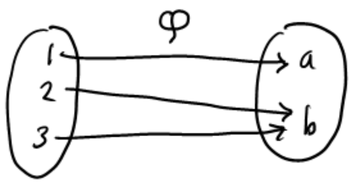
The condition $(x, y) \in \varphi$ means φ sends x to y, and is abbreviated as $\varphi(x) = y$. Intuitively we think of φ as a "rule" sending X to Y.

The set X is called the domain of φ.
The set Y is called the codomain of φ
The range of φ is the set $\{\varphi(x) \mid x \in X\}$ (i.e. set of all "outputs.")

Example $X = \{1, 2, 3\}$ $Y = \{a, b\}$
$\varphi = \{(1, a), (2, b), (3, b)\}$ ⇝ $\begin{cases} \varphi(1) = a \\ \varphi(2) = b \\ \varphi(3) = b \end{cases}$

Ways of drawing φ:



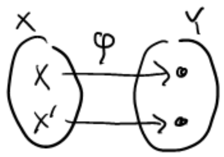


Definitions

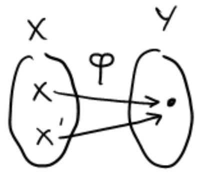
- $\varphi: X \to Y$ is <u>one-to-one</u> if $\varphi(x) = \varphi(x') \Rightarrow x = x'$

  i.e. $x \neq x' \Rightarrow \varphi(x) \neq \varphi(x')$

One-to-one means unequal elements are sent to unequal elements

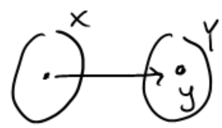


and <u>never</u>



- $\varphi: X \to Y$ is <u>onto</u> if whenever $y \in Y$, there is at least one $x \in X$ with $\varphi(x) = y$

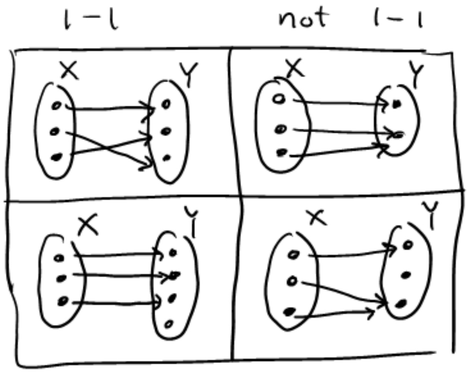
Onto means "every $y \in Y$ has an arrow pointing to it i.e. "Every $y \in Y$ is touched by φ."



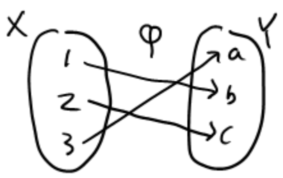
Examples:

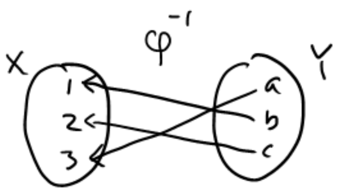
onto

not onto



<u>Recall</u> a 1-1 and onto function $\varphi: X \to Y$ has an inverse $\varphi^{-1}: Y \to X$ satisfying $\varphi^{-1}(\varphi(x)) = x$ and $\varphi(\varphi^{-1}(y)) = y$.





Cardinality

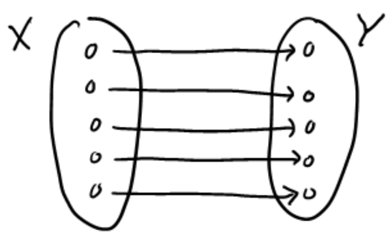
Roughly, the cardinality $|X|$ of a set X is the number of elements in X

$A = \{a, b, c\}$ $\quad |A| = 3$ $\quad |A \times B| = 6$

$B = \{0, 1\}$ $\quad |B| = 2$ $\quad |\mathcal{P}(A)| = 8$

This idea is more interesting and subtle than you might expect especially when the sets are infinite. For example, how do $|\mathbb{Z}|$, $|\mathbb{Z}^+|$, $|\mathbb{Q}|$ and $|\mathbb{R}|$ compare?

<u>Definition</u> Two sets X and Y have the <u>same cardinality</u>, written $|X| = |Y|$ if there exists a one-to-one and onto function $\varphi: X \to Y$.



Now see slides concerning cardinality.
[Some are too detailed to write on board]

A few words about one of your homework problems.

<u>Section 0, Exercise 18</u>

Suppose A is <u>any</u> set (finite or infinite)
and let $B = \{0, 1\}$.

Denote by B^A the set of all functions $f : A \to B$

Show $|B^A| = |\mathcal{P}(A)|$

<u>Example</u> $A = \{a, b\}$

$B^A = \{$  $\}$

$\mathcal{P}(A) = \{ \phi, \quad \{a\}, \quad \{b\}, \quad \{a, b\} \}$

Note that $|B^A| = 4 = |\mathcal{P}(A)|$
The question is asking you to show that $|B^A| = |\mathcal{P}(A)|$ always holds.

<u>Strategy</u>: Construct a one-to-one onto function $\varphi : B^A \to \mathcal{P}(A)$
i.e. if $f \in B^A$, then $\varphi(f) \in \mathcal{P}(A)$, i.e. $\varphi(f) \subseteq A$.

You will need to:
1. Define φ.
2. Show φ is 1-1
   i.e. show that if $f, g \in B^A$ and $f \neq g$
   then $\varphi(f) \neq \varphi(g)$
3. Show φ is onto
   i.e. given and $X \in \mathcal{P}(A)$ $(X \subseteq A)$
   there is a function $f : A \to B$
   with $\varphi(A) = X$