MATH 504 Day 1

This is a course in Abstract Algebra

The algebra you studied in high school (and the algebra you will teach) was concrete and predictable. Variables represented numbers and numbers behaved in predictible ways

$$11 + 3 = 14$$

Later things got more complex and you moved away from numbers and the familiar operations of $+, -, \cdot, \div$, etc,

Calculus $\quad f\circ g = h \qquad f\circ g \neq g\circ f$

Linear Algebra $\quad AB \neq BA$

Some old rules of algebra began to break down. In fact, old rules break down in even very ordinary settings like adding hours:

$$11 + 3 = 2$$

The goal of abstract algebra is to develop systems that are general enough to deal with exotit situatios like these. There are great advantages to doing this, as the scope of algebra is widened greatly.

Studying abstract algebra can also greatly increase your understanding of elementary (i.e. high school) algebra.

We will begin with Section 0, which describes the most basic ideas that you will need, and on which all of our further work will be based

This is a quick review of material you learned in MATH 300

Section 0 Sets and Relations

The concept of a set underlies everything we do in this course. In abstract algebra you are not necessarily dealing with numbers in a number system, but rather elements in a set.

Definition A set is a collection of things. The things in a set are called its elements. Sets are often denoted by upper-case letters.

$A = \{2, 3, 7\} \qquad 2 \in A, \ 3 \in A, \ 7 \in A, \ 8 \notin A$

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in B \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin B$

$C = \{ \{1,2\}, 3 \}$ $\{1,2\} \in C$, $3 \in C$, $2 \notin C$.

Some sets come up so often that they have special names.

<u>Empty Set</u> $\phi = \{ \}$

<u>Integers</u> $\mathbb{Z} = \{ \ldots -2, -1, 0, 1, 2, 3, \ldots \} = \{0, 1, -1, 2, -2, 3, -3, \ldots \}$

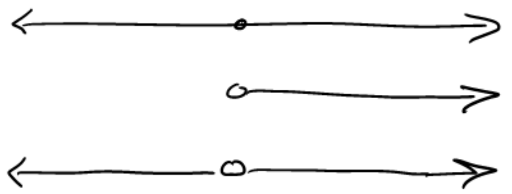
$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots \} = \{ x \in \mathbb{Z} \mid x > 0 \}$

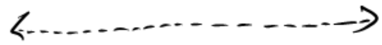
$\mathbb{Z}^* = \{ \ldots -3, -2, -1, 1, 2, 3, \ldots \} = \{ x \in \mathbb{Z} \mid x \neq 0 \}$

<u>Real numbers</u> $\mathbb{R}$

$\mathbb{R}^+ = \{ x \in \mathbb{R}, \ x > 0 \}$

$\mathbb{R}^* = \{ x \in \mathbb{R}, \ x \neq 0 \}$



<u>Rationals</u> $\mathbb{Q} = \{ x \mid x = \frac{a}{b}, \ a, b \in \mathbb{Z}, \ b \neq 0 \}$ 

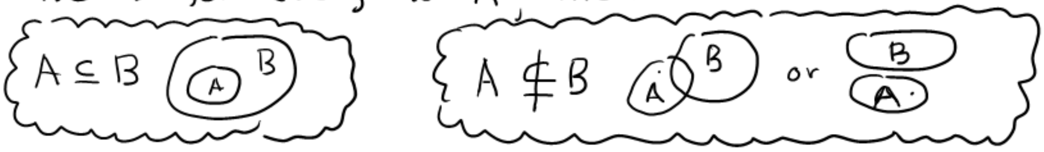
$\mathbb{Q}^+ = \{ x \in \mathbb{Q} \mid x > 0 \}$

$\mathbb{Q}^* = \{ x \in \mathbb{Q} \mid x \neq 0 \}$

There will be other special sets, introduced as needed.

Subsets

$A \subseteq B$ means for every $x \in A$, then also $x \in B$.



Note $A \nsubseteq B$ means there is an $x \in A$ with $x \notin B$

<u>Example</u> $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ <u>Example</u> $\{a, c\} \subseteq \{a, b, c\}$
$\{a\} \subseteq \{a, b, c\}$

<u>Important Fact</u> If B is <u>any</u> set, then $\phi \subseteq B$.

<u>Reason</u> Could it be that $\phi \nsubseteq B$?

If this were so then you could find an $x \in \phi$ with $x \notin B$
But $x \in \phi$ is impossible. Thus $\phi \nsubseteq B$ is untrue. Hence $\phi \subseteq B$

Thus $\phi \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$\phi \subseteq \{a, b\}$

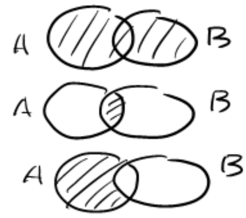
Be careful: $\left\{ \begin{array}{l} \phi \notin \{a, b\} \\ \phi \in \{a, b, \phi\} \\ \phi \subseteq \{a, b, \phi\} \end{array} \right.$

New Sets From Old

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Difference: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$



Power Sets

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}$$

Example $A = \{1, 2, 3\}$

$$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Cartesian Product

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

Example $\{a, b\} \times \{1, 2, 3\} = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

Example $\mathbb{Z}^+ \times \mathbb{Z}^+ = \{ (1,1), (1,2), (1,3) \ldots$
$(2,1)\ (2,2)\ (2,3) \ldots$
$(3,1)\ (3,2), (3,3) \ldots \}$

Example $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\} = \mathbb{R}^2$ (plane)

Relations

A relation between A and B is a subset $\mathcal{R} \subseteq A \times B$

Condition $(x,y) \in \mathcal{R}$ is abreviated $x \mathcal{R} y$ (read "x relates to y")

Example $A = \{a, b\}$ $B = \mathcal{P}(\{a,b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$$\mathcal{R} = \{(a, \{a\}), (b, \{b\}), (a, \{a,b\}), (b, \{a,b\})\} \subseteq A \times B$$

$$\left.\begin{array}{l} a \mathcal{R} \{a\} \\ b \mathcal{R} \{b\} \\ a \mathcal{R} \{a,b\} \\ b \mathcal{R} \{a,b\} \end{array}\right\} \mathcal{R} \text{ is } \in.$$

Example $\mathcal{R} = \{(x,y) \mid x, y \in \mathbb{R}, x - y \in \mathbb{R}^+\} \subseteq \mathbb{R} \times \mathbb{R}$

$$\left.\begin{array}{ll} (3, 2) \in \mathcal{R} & 3 \mathcal{R} 2 \\ (\pi, 3) \in \mathcal{R} & \pi \mathcal{R} 3 \\ (-3, -5) \in \mathcal{R} & -3 \mathcal{R} -5 \end{array}\right\} \mathcal{R} \text{ is } >$$

etc.

Important Point Relations such as $\in$ and $>$, etc can be described this way.