Section 2 Solutions

(2) \((a \ast b) \ast c = b \ast c = a\)  
\(a \ast (b \ast c) = a \ast a = a\)

Even though we’ve shown that \((a \ast b) \ast c = a \ast (b \ast c)\), that’s no guarantee that the operation \(\ast\) is associative. We would have to show \((x \ast y) \ast z = x \ast (y \ast z)\) for all possible values of \(x, y\) and \(z\). In fact, note that \((d \ast a) \ast b = b \ast b = c\) is unequal to \(d \ast (a \ast b) = d \ast b = c\), so \(\ast\) is NOT ASSOCIATIVE.

(4) The operation \(\ast\) is NOT COMMUTATIVE because, for instance, \(e \ast b = b\) but \(b \ast e = c\).

(6) Suppose the following partial table is for an associative binary operation on \(S = \{a, b, c, d\}\).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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</tbody>
</table>

The missing line should give the values of \(d \ast x\) for the various \(x\). To fill in this line, use the fact that the table gives \(c \ast b = d\), together with the fact that \(\ast\) is associative:

\[
\begin{align*}
(d \ast a) &= (c \ast b) \ast a = c \ast (b \ast a) = c \ast b = d \\
(d \ast b) &= (c \ast b) \ast b = c \ast (b \ast b) = c \ast a = c \\
(d \ast c) &= (c \ast b) \ast c = c \ast (b \ast c) = c \ast c = c \\
(d \ast d) &= (c \ast b) \ast d = c \ast (b \ast d) = c \ast d = d
\end{align*}
\]

Thus the completed table is as follows

<table>
<thead>
<tr>
<th></th>
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</tbody>
</table>

(10) Consider the binary operation on \(\mathbb{Z}\) defined as \(a \ast b = 2^{ab}\).

This is COMMUTATIVE because \(a \ast b = 2^{ab} = 2^{ba} = b \ast a\) for all \(a, b \in \mathbb{Z}\).

This is NOT ASSOCIATIVE because, in particular

\[
\begin{align*}
0 \ast (1 \ast 2) &= 0 \ast (2^{1\cdot 2}) = 0 \ast 4 = 2^0 \cdot 4 = 2^0 = 1 & \\
(0 \ast 1) \ast 2 &= (2^{0\cdot 1}) \ast 2 = 1 \ast 2 = 2^{1\cdot 2} = 2^2 = 4
\end{align*}
\]

(36) Suppose \(\ast\) is an associative binary operation on a set \(S\), and \(H = \{a \in S | a \ast x = x \ast a\ \text{for all} x \in S\}\). Show \(H\) is closed under \(\ast\).

Proof. Suppose that \(a\) and \(b\) are two arbitrary elements of \(H\). To show \(H\) is closed, we must show that \(a \ast b \in H\). And to show \(a \ast b\) is in \(H\) we must show \(a \ast b\) satisfies the requirement for being in \(H\), that is we must show \((a \ast b) \ast x = x \ast (a \ast b)\) for every element \(x\) in \(S\).

Let \(x\) be an arbitrary element of \(S\). The fact that \(a\) and \(b\) are in \(H\) means

\[
\begin{align*}
a \ast x &= x \ast a \\
b \ast x &= x \ast b
\end{align*}
\]

Using (1) and (2) together with associativity of \(\ast\), we deduce

\[
(a \ast b) \ast x = a \ast (b \ast x) = a \ast (x \ast b) = (a \ast x) \ast b = (x \ast a) \ast b = x \ast (a \ast b).
\]

Thus \((a \ast b) \ast x = x \ast (a \ast b)\), which means \(a \ast b \in H\), so \(H\) is closed.