Math 490 Projects
for Mathematics Majors Planning to Teach 9-12 Mathematics

For the projects outlined below you are provided with a list of possible references. Most of these books and articles can be found in VCU’s libraries but some will need to be requested through interlibrary loan. These are designed to help you start researching the project but they should not be the only references you depend on as you work on the project. Your finished product should include references you find as you do the research.

1. **Incommensurability**
   A pentagram is the figure formed by the diagonals of a regular pentagon. A pentagram has segments that are incommensurable (i.e. there are ratios of lengths of segments in a pentagram that are irrational). It has been said that the pentagram was the first geometric figure proved to have incommensurable segments. Research whether or not this is true as well as the history of incommensurability and the pentagram. A proof of the incommensurable segments of a pentagram can be found in the College Mathematics Journal 11(5), 1980, p. 312-316. Study and explain the proof. Use the central idea of the proof to show that the leg and hypotenuse of an isosceles right triangle are incommensurable. Explain how this shows that $\sqrt{2}$ is irrational. Find and include in your paper the traditional proof that $\sqrt{2}$ is irrational.


2. **Cardano-Tartaglia Method for Solving Cubic Equations**
   Research the history of cubic polynomials and early attempts at finding solutions to these types of equations. One early method for solving cubic polynomial equations is called the Cardano-Tartaglia method. Given a cubic equation of the form $x^3 + ax^2 + bx + c = 0$ where $a, b, c$ are real numbers show that the change of variables $x = y - \frac{a}{3}$ reduces the original equation to a cubic equation $y^3 + py + q = 0$. Notice that this equation has no quadratic term. Show that the change of variables $y = z - \frac{p}{3z}$ reduces the second equation to a quadratic equation of the form $(z^3)^2 + qz^3 - \frac{p^3}{27} = 0$. If this last equation is solved for $z$, the six solutions lead back to solutions of the original equation in $x$. Apply the Cardano-Tartaglia method to a cubic equation. Use a computer or calculator with

Some of these projects were adapted from problems found in *Mathematics for High School Teachers: An Advanced Perspective* by Usiskin, Peressini, Marchisotto, & Stanley
symbolic algebra capabilities to solve your cubic equation and compare the answers the
technology provides with the answers provided by the Cardano-Tartaglia method. The
cubic equation $x^3 - 15x - 4 = 0$ is called the Bombelli cubic equation. Research
Bombelli’s role in the analysis and study of cubic equations. Apply the Cardano-
Tartaglia method to the Bombelli cubic equation and discuss the results.

Mathematica*, 107-112.
Monthly*, 40 (7), 411-412.

3. **Leibniz Segments**

Gottfried Leibniz was the first to use the word “function” to describe a mathematical
situation. In 1694 he used the word “function” to describe six line segments associated
with a specific point on a curve. The six line segments are segments $\overline{OQ}$, $\overline{PQ}$, $\overline{PT}$, $\overline{PN}$,
$\overline{TQ}$, and $\overline{NQ}$ in the diagram to the right. Given a curve $C$ and a point $P$
on $C$, let $T$ be the point of
intersection of the tangent line to $C$
at $P$ with the horizontal axis, let $N$ be
the point of intersection of the
normal line to $C$ at $P$ with the
horizontal axis, and let $Q$ be the foot
of the perpendicular from the
horizontal axis through $P$. Leibniz
introduced the following six
functions of the point $P$ on $C$.

a. The **abscissa** at $P$ is the line segment $\overline{OQ}$.
b. The **ordinate** at $P$ is the line segment $\overline{PQ}$.
c. The **tangent** at $P$ is the line segment $\overline{PT}$.
d. The **normal** at $P$ is the line segment $\overline{PN}$.
e. The **subtangent** at $P$ is the line segment $\overline{TQ}$.
f. The **subnormal** at $P$ is the line segment $\overline{NQ}$. 
Research these six functions and Leibniz use of the word “function”. Provide the formal definitions and some history. Leibniz studied the fact that the slope of the tangent line at P is equal to three different ratios of these six functions. Research how Leibniz studied this and what he found. What are these ratios? The geometric mean (aka mean proportional) of two positive numbers \( x \) and \( y \) is \( \sqrt{xy} \). Prove that the length of the ordinate at P is the geometric mean of the length of the subtangent and the length of the subnormal. Apply the Leibniz segments to a specific equation. Given the parabola \( 2x = y^2 + 3 \), find the length of all six line segments for the point \( P = (6, 3) \). Provide a sketch of the parabola and the six line segments. The formal definition of parabola involves a focus and a directrix. Find the formal definition of parabola and research some of the history of parabolas. For the general parabola, \( y^2 = 2px \), explain why the focus is at \( (\frac{p}{2},0) \) and the directrix is the line \( x = -\frac{p}{2} \). For a general point \( P = (x_1,y_1) \) on the parabola \( y^2 = 2px \), find the x-coordinates of the points T, Q, and N in terms of \( y_1 \).


4. **Viete’s Theorem**

Quadratic equations date back to the ancient Babylonians. There is evidence that they used quadratic equations to calculate prices for selling items at market and the origins of the quadratic formula can be found on Babylonian tablets. The ancient Babylonians looked for a way to find equations (some of which were quadratic) for a predetermined solution set. In particular, a Babylonian tablet dated around 1760 B.C. contains the following problem: Find two numbers whose sum is 10 and whose product is 18. The solution provided on the tablet can be generalized to yield a solution to any quadratic equation in one variable. Start by researching the history of quadratic equations. Provide details from the Babylonian and Egyptian civilizations as well as more modern uses of quadratic equations. Consider the equations posed by question found on the Babylonian tablet: \( m + n = 10 \) & \( mn = 18 \). Construct a quadratic equation from these two equations. Explain the similarities between the quadratic equation and the two equations from which it was derived. This result leads to Viete’s Theorem. This theorem is named after the French mathematician who discovered it, Francois Viete (or Vieta) (1540 – 1603). It applies to a special case of quadratic equations of the form \( x^2 + bx + c = 0 \). Research the history of Francois Viete and the theorem. Find the theorem and prove it. (Hint: Start with the fact that \( m \) and \( n \) are distinct solutions of the quadratic equation \( x^2 + bx + c = 0 \), so \( m^2 + bm + c = 0 \) and \( n^2 + bn + c = 0 \).) Apply Viete’s Theorem to the general quadratic
equation $ax^2 + bx + c = 0$. This results in a Corollary of Viète’s Theorem. State it. If $m$ and $n$ are solutions to the quadratic equation $x^2 + bx + c = 0$ and $m + n = -b$, then the average of $m$ and $n$ is $\frac{b}{2}$. Therefore, let $m = \frac{b}{2} + x$ and $n = \frac{b}{2} - x$. (Do you agree?) Substitute these values into the second part of Viète’s Theorem and solve for $x$. What is the result? How does this result relate to the quadratic formula? Use the completing the square process to prove the quadratic formula provides solutions for the general quadratic equation $ax^2 + bx + c = 0$.


5. **Fundamental Theorem of Algebra**

In 1629, Albert Girard (1595 – 1632) stated that a polynomial equation of degree $n$ has $n$ solutions. He did not provide proof for this statement, but his work is the earliest reference to the Fundamental Theorem of Algebra. Descartes also studied solutions to polynomial equations and extended the work of Girard. Gauss (1777 – 1855) published the first proof of the Fundamental Theorem of Algebra in 1799. He wrote a total of five proofs for the theorem in all. His first proof did not outline the possibility that the polynomial could have complex coefficients. Research the history of Girard, Descartes, and Gauss in relation to the Fundamental Theorem of Algebra. State the Fundamental Theorem of Algebra. Find and explain two of Gauss’ proofs of the theorem. Explain the significance of the Fundamental Theorem of Algebra to finding solutions to polynomial equations.


6. **Quadrature**

The ancient Greeks were interested in the problem of quadrature. Quadrature is the ability to construct a square with the same area as a given figure. If a square can be constructed in this manner, then you know the area of the original figure in square units. This was an important concept for the ancient Greeks. Research the history of quadrature. Research when this was first used and who was instrumental in developing the concept. Apply quadrature to a rectangle with dimension $x$ units by $y$ units. In other words, explain how to construct a square with the same area as the rectangle with area $xy$. Then apply the concept to a triangle. Construct a square with the same area as a triangle with base $b$ and height $h$. You might find it helpful to first construct a rectangle with the same area as a triangle with base $b$ and height $h$. Can quadrature be applied to a trapezoid? Explain. Research the classic problem applying quadrature to a circle (i.e. squaring a circle). Provide details of the history of this problem. In the 19th century, it was proven that it is not possible to square a circle with compass and straightedge. Explain why. One way to approach this problem is to start with a circle, then circumscribe a square about the circle and inscribe another square inside the circle. Construct a third square whose side length is the mean of the side lengths of the circumscribed and inscribed squares. Compare the area of the third square to the area of the circle. Discuss whether this method is helpful in squaring a circle. Research whether this problem can be solved if tools other than compass and straightedge can be used.

A locus is a set of points that satisfy a certain condition or set of conditions. Locus problems have been studied by mathematicians for centuries. Greek scholars such as Euclid, Apollonius of Perga, and Pappus of Alexandria were some of the first to look for a set of points that satisfy particular conditions. Research the history of locus problems. Provide examples of some of the standard locus problems studied throughout history. One classic locus problem involves an animal on a leash that is tied to a side of a building. Here is one version of this classic problem:

A dog on a leash is tied to a rectangular-shaped barn that is 20 meters long and 10 meters wide. The leash is 8 meters long and is fastened 6 meters from the end of the longer side of the barn. Sketch the region in which the dog can roam and find the area of that region.

Solve this problem. Then, without changing any of the numerical values in the problem, determine where the leash should be tied on the building to maximize the area the dog can roam. Now, without changing any other numerical values in the problem, let the length of the leash be \(x\) meters. Assume the barn is in the center of a flat field that extends infinitely in all directions. Let \(A(x)\) be the area function that represents the region in which the dog can roam. Answer the following questions:

a. For what values of \(x \leq 30\) does the function change?
b. Provide an algebraic statement for \(A(x)\) for all values of \(x \leq 30\). Graph the functions.
c. On the interval \(0 \leq x \leq 30\) is \(A\) continuous? Explain.
d. On the interval \(0 \leq x \leq 30\) is \(A\) differentiable? Explain.
e. As \(x \to \infty\), what does \(A\) approach?

Answer questions a – e for the same conditions except the leash is tied to the upper right corner of the barn. How do the answers change if the leash is tied to one of the other corners of the barn? Make one final change to the problem. Let all of the numerical values be the same as the original problem statement (including the length of the leash)
except let the dog be tethered to the long side of the barn $d$ meters from the upper right corner of the barn where $0 \leq d \leq 20$. Let $B(d)$ be the area function that represents the region in which the dog can roam. Give a formula for $B(d)$ in terms of $d$. Graph $B$ and explain what the graph shows. Based on your detailed analysis of this problem, discuss the value of locus problems to real-world situations. Explain what you learned from working these problems.


8. The Erdos-Mordell Theorem
Paul Erdos was one of the most prolific mathematicians that ever lived. He is arguably one of the greatest mathematicians of the 20th century. Research the history of Paul Erdos. Describe the fields of mathematics he influenced, the variety of papers he published, and the mathematical problems he posed for others to solve over the years. In 1935, Erdos published the following conjecture in American Mathematical Monthly:

If, from a point $P$ inside a given triangle $ABC$, perpendiculars $PD$, $PE$, and $PF$ are drawn to its sides, then $PA + PB + PC \geq 2(PD + PE + PF)$. Equality holds if and only if triangle $ABC$ is an equilateral triangle. Louis Mordell and David Barrow proved the theorem separately within a year after the conjecture was published. Several other proofs of what is now called the Erdos-Mordell theorem have appears in American Mathematical Monthly over the last sixty years (Bankoff (1958), Avez (1993), Komornik (1997)). Oppenheim (1961) published an American Mathematical Monthly article that contained some inequalities similar to the one posed in the Erdos-Mordell Theorem. These proofs involve some interesting geometric terms including the centroid of a triangle, the circumcenter of a triangle, the pedal point of a triangle, and a cyclic quadrilateral. Find definitions of these terms (and any others you discover along the way) and provide diagrams for each. You might find using a geometry software tool like Geometer’s Sketchpad useful. Study all five proofs of the Erdos-Mordell theorem and the inequalities posed by Oppenheim. Explain the similarities and differences in the proofs. Which one do you prefer and why? Explain Oppenheim’s inequalities and how they are useful.
The Four Color Theorem

One area of mathematics that has some interesting problems and applications is graph theory. Study the basic concepts of graph theory in preparation for this project. Focus on the definition of terms and some basic theorems. One interesting problem was posed by Francis Guthrie in 1852. When coloring a map of the counties of England, he noticed it only took four colors to ensure that no adjacent counties were the same color. He wondered if this was always the case. This became known as the Four Color Conjecture until it was official proved to be true in 1976 by Appel and Haken. Research the life of Francis Guthrie. Which other mathematicians became involved in working on this problem over the years? The Three Color Theorem states that more than three colors are required to color some maps so that no two adjacent regions are the same color. The Two Color Theorem states that if a map is made up of lines which cross each other and do not stop when they meet another line, then two colors can be used to color the map so that no two adjacent regions are the same color. Research the history of the Two and Three Color Theorems. How do they relate to the Four Color Theorem? Research the evolution of the Four Color Conjecture to the Four Color Theorem. Along the way, mathematicians proved a Five Color Theorem and a Six Color Theorem. Include discussion of these theorems in your paper. What was
their significance in the journey to four colors? The Four Color Theorem was one of the first to be solved by computer. How did they do it? At least one other proof has been published since Appel and Haken’s proof of the Four Color Theorem. Research this other proof. How does it compare to the proof of Appel and Haken?


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10. **Arbelos**

Arbelos is the Greek word for “shoemaker’s knife”. It refers to the shape of the region enclosed in the large semicircle but not inside the two smaller semicircles in the figure to the right which looks like the knife used by cobbler in ancient Greece. Archimedes was one of the first mathematicians to study the mathematical properties of the arbelos. Research the life of Archimedes and the history of the word “arbelos”. This shape is the upper half of a figure that consists of three circles. Two smaller circles are outside each other, but inside a third, larger circle. Each of the three circles is tangent to the other two and their centers are along the same straight line. In the diagram provided, the middle notch can be located anywhere along the diameter of the large circle. This shape has some very interesting properties. Research the properties of this shape. You might find the use of a geometry software program like Geometer’s Sketchpad useful in constructing the arbelos and other shapes in this project. If the diameter of the large semicircle is 1 unit and the diameter of the left inner semicircle is d while the diameter of the right inner semicircle is...
d – 1, what is the arc length of the bottom of the arbelos? What is the area of the arbelos? If a perpendicular line to the diameter is constructed at the point of tangency of the two inner circles, the circles inscribed in each side of the arbelos are called Archimedes circles. Research how Archimedes circles are constructed and some interesting properties of Archimedes circles. Make note of the fact that the circumcircle of the Archimedes circles has the same area as the original arbelos. Another circle inside the arbelos is called the Apollonius circle. Research this circle and its interesting properties.

A set of tangent circles inside the arbelos is called the Pappus chain. Research the Pappus chain and its interesting properties.


