"Shutting up like a telescope": Lewis Carroll's "Curious" Condensation Method for Evaluating Determinants

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As every algebra student knows, given a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

its determinant $\det A$ (or $|A|$) is calculated by finding the difference between the products of the diametrically opposed entries in the matrix. That is, $\det A = ad - bc$. Given any $n \times n$ matrix A, its determinant will provide useful information, both algebraic and geometric. For example, geometrically, the row entries of A define the edges of a parallelepiped in n-dimensional space, of which the volume is simply the value of $\det A$. Algebraically, the matrix A represents the coefficients of a system of n linear equations in n unknowns. The value of the determinant of A determines whether or not this system is solvable. In particular, if $\det A$ is nonzero, we know that the inverse matrix exists, and this in turn implies the solvability of the system of linear equations represented by matrix A.

Determinants emerged gradually during the 18th century through the theory of equations in the work of Leibniz, Maclaurin, Cramer, and Laplace. By the 19th century, the subject had become a mathematical area of increasing significance. Gauss (who invented the name *determinant* [**16**, vol. 1, p. 64]), Cauchy, and Cayley all produced important results on the subject, and in 1841, the German mathematician Carl Jacobi [**12**], [**13**], [**14**] published three major papers which finally brought the subject into the mathematical mainstream.

As we have seen, 2×2 determinants can be calculated very easily, but as n increases, computations become more time-consuming. The standard method of computing a determinant is to break it down into more determinants of lower degree by taking the product of each row or column entry and the determinant of its complementary minor and then alternately adding and subtracting the results. This method was devised by Laplace [**15**] in 1772 and is nowadays based on the following definitions and a theorem, which still bears his name.

Given an $n \times n$ matrix A, a *minor* is any $(n-m) \times (n-m)$ matrix formed by deleting m rows and m columns from A. A *complementary minor* is the resulting $m \times m$ matrix diagonally adjacent to the minor matrix. A *consecutive minor* is one in which the remaining rows and columns in the minor were adjacent in the original matrix. Finally, $a'_{ij} = (-1)^{i+j}\det[A_{ij}]$ is the *cofactor* of a_{ij} in A, where $[A_{ij}]$ denotes the minor matrix obtained by deleting the ith row and jth column in A.

For example, consider the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & -3 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & 2 & -1 \\ 0 & -2 & 3 & 1 \end{pmatrix}.$$

If we let $m = 2$ and delete the second and third rows and columns, we obtain the minor matrix $\begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$ and the complementary minor $\begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}$, which is also consecutive.

Theorem 1. *If $A = [a_{ij}]$ is an $n \times n$ matrix, the determinant of A,*

$$\begin{aligned} \det A &= a_{r1}a'_{r1} + a_{r2}a'_{r2} + \cdots + a_{rn}a'_{rn} \qquad (\textit{for } 1 \le r \le n) \\ &= a_{1s}a'_{1s} + a_{2s}a'_{2s} + \cdots + a_{ns}a'_{ns} \qquad (\textit{for } 1 \le s \le n). \end{aligned}$$

Continuing with our example from above, using the cofactors of the first row of matrix A, we get

$$\begin{aligned} \det A &= a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} \\ &= 2\begin{vmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \\ 0 & -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & 2 \\ 0 & -2 & 3 \end{vmatrix} \\ &= 2(-7) - 1(-4) - 1(5) + 3(7) \\ &= 6. \end{aligned}$$

This method quickly gets long and laborious as the size of the matrix increases. In fact, to find the determinant of a 5×5 matrix a considerable amount of calculation is required, involving five 4×4 determinants, which break down into twenty 3×3

determinants, or *sixty* 2×2 determinants, or *120* actual multiplications, plus all the additions required, before you get to the final answer. Another popular method is to use elementary row operations to produce a triangular matrix, whose determinant is simply the product of the diagonal entries. However, this method can also involve a considerable amount of work.

Of course these days, with the help of packages like *Mathematica*, *Maple*, or *Matlab*, determinants can be easily calculated in a matter of seconds. But before the 20th century, mathematicians had no such luxury, nor indeed does everyone today. But there is another method, first introduced in 1866 and widely ignored since, which can simplify the work involved in calculating determinants of large matrices considerably, and which, we believe, can still be of interest to today's students.

The condensation method

The algorithm is one of considerable computational simplicity, achieved by restricting itself entirely to the calculation of 2×2 determinants. We first need another definition:

Given an $n \times n$ matrix A, with $n \geq 3$, the *interior* of A, or int$A$, is the $(n-2) \times (n-2)$ consecutive minor that results when the first and last rows and columns of matrix A are deleted.

The method consists of the following steps:

- Remove all zeros from the interior of A, using elementary row and column operations. Call this matrix $A^{(0)}$.
- Find the determinant of every 2×2 consecutive minor in $A^{(0)}$ to form a new $(n-1) \times (n-1)$ matrix $A^{(1)}$.
- Now find the determinant of every 2×2 consecutive minor in $A^{(1)}$ to produce an $(n-2) \times (n-2)$ matrix. Then divide each term by the corresponding entry in the interior of matrix $A^{(0)}$. This will give a new matrix $A^{(2)}$.
- In general, given the matrix $A^{(k)}$, compute a new $(n-k-1) \times (n-k-1)$ matrix made up of the determinants of the 2×2 consecutive minors of $A^{(k)}$. To produce $A^{(k+1)}$, divide each of these entries by the corresponding entry in the interior of $A^{(k-1)}$.
- Continue repeating the previous step, 'condensing' the matrix until a single number is obtained. This will be $\det A$.

As an illustration of this method, we continue with our example. Consider again the 4×4 matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & -3 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & 2 & -1 \\ 0 & -2 & 3 & 1 \end{pmatrix} = A^{(0)}.$$

Finding all determinants of 2×2 consecutive minors, we 'condense' the matrix into the 3×3 matrix

$$A^{(1)} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -5 & 1 & 9 \\ 7 & -7 & -3 \\ -6 & 7 & 5 \end{pmatrix}.$$

This matrix is, in turn, similarly condensed into the matrix

$$\begin{pmatrix} \begin{vmatrix} -5 & 1 \\ 7 & -7 \end{vmatrix} & \begin{vmatrix} 1 & 9 \\ -7 & -3 \end{vmatrix} \\ \begin{vmatrix} 7 & -7 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} -7 & -3 \\ 7 & 5 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 28 & 60 \\ 7 & -14 \end{pmatrix}.$$

We now divide each entry of this matrix by the corresponding term in the interior of matrix $A^{(0)}$ to obtain

$$A^{(2)} = \begin{pmatrix} -14 & 20 \\ 7 & -7 \end{pmatrix}.$$

The determinant of matrix $A^{(2)}$ is -42, which, when divided by -7, the interior term of matrix $A^{(1)}$, gives 'matrix' $A^{(3)}$ containing as its only entry the correct answer of 6.

But why does this method work, and where did it come from?

Enter Lewis Carroll

The inventor of this ingenious method was none other than the Reverend Charles Lutwidge Dodgson (1832–1898), a Church of England clergyman who earned his living as a mathematics lecturer at Christ Church, Oxford. But it was through his hobby of writing children's books, puzzles, and verses under the pseudonym "Lewis Carroll" that he became best known, particularly after the publication of his most successful work, *Alice in Wonderland*, in 1865. In addition to this, he also wrote and published a considerable number of books on mathematical subjects under his real name, including *A Syllabus of Plane Algebraical Geometry* (1860) and *Euclid and his Modern Rivals* (1879). His persona as a vibrant and creative children's author was in marked contrast to his more sober mathematical publications. Indeed, as a mathematician, while certainly painstaking and methodical, Dodgson was far from famous or first-rate, being rather conservative in his tastes and approach.

The best illustration of this is his dogmatic and reactionary insistence on the superiority of Euclid's *Elements* as a mode of teaching geometry. It is perhaps unfortunate that much of Dodgson's mathematical work was devoted to matters arising from the study of Euclid's *Elements* and Euclidean geometry, for the simple reason that the *Elements* is far less widely used in mathematics teaching today than in Dodgson's day, when it was virtually ubiquitous. Consequently, none of Dodgson's geometrical work is used nowadays.

Although more of a recreational mathematician than a serious researcher, Dodgson did make a number of small contributions to the subject, most of which are largely unknown to today's mathematicians [**2**]. Some of these would nowadays be classified as part of game theory, although that subject did not really come into existence until after the Second World War. In his analysis of tennis tournaments as well as his study of the theory of elections and voting patterns, Dodgson's work pre-dated the birth of this topic by over half a century [**3**]. His work in logic was also progressive. Although his published work was largely recreational in style, his (unpublished) work on symbolic logic was ahead of its time in several respects, such as the use of trees and truth tables to solve specific logic problems [**5**].

Yet, somewhat paradoxically, we are told that Dodgson's innate originality forms the chief obstacle to his mathematical work. "He read comparatively little of the works of other mathematicians or logicians, preferring to develop his theories out of his own mind. This method had its advantages, no doubt, yet it not only gave him a lot of unnecessary trouble but deprived him of the chance of escaping avoidable mistakes. In fact, he handled scientific matters in the same way as he dealt with conversational language, and the method was never likely to produce—nor did it produce—a mathematical achievement of comparable value, in its own line, to *Alice in Wonderland*." [**9**, p. 132]

Interestingly, Dodgson's most original mathematical research was undertaken in the mid-1860s, at roughly the same time as the publication of *Alice In Wonderland*. As he noted in his diary on 27 February 1866: "Discovered a process for evaluating arithmetical Determinants by a sort of condensation and proved it up to 4^2 terms" [**1**, p. 334]. This method was published in his first and, as it turned out, his last research paper, in the *Proceedings of the Royal Society of London* later that year and introduced his 'condensation' method for evaluating large determinants. It was later incorporated in his only algebra book, *An Elementary Treatise on Determinants, with their application to simultaneous linear equations and algebraical geometry*, published in 1867.

Jacobi's theorem

As Dodgson himself acknowledged, his method was an application of a "well-known theorem in determinants" [**7**, p. 152], but he said nothing about where it came from. In fact, the theorem on which his method was based can be traced back to Jacobi, although special cases of it had been derived by Lagrange in 1773, Desnanot in 1819, and Cauchy and Minding in 1829.[1] Jacobi's first statement and proof of the theorem appeared in a paper in *Crelle's Journal* in 1833, with the result being repeated and elaborated in further papers of 1835 and 1841, also in *Crelle* [**10**], [**11**], [**12**].

In his *Elementary Treatise on Determinants*, Dodgson stated Jacobi's theorem as follows:

> *If there be a square Block of the nth degree, and if in it any Minor of the mth degree be selected: the Determinant of the corresponding Minor in the adjugate Block is equal, in absolute magnitude, to the product of the* $(m-1)$*th power of the Determinant of the first Block, multiplied by the Determinant of the Minor complemental to the one selected.* [**8**, p. 25]

To understand this theorem in its most general form, we need one more definition:

[1] One special case of Jacobi's theorem is given by Bressoud [**6**, pp. 112–113].

The *adjugate* of an $n \times n$ matrix A is

$$A' = \begin{pmatrix} a'_{11} & a'_{12} & \cdot & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & \cdot & a'_{nn} \end{pmatrix},$$

where $a'_{ij} = (-1)^{i+j} \det[A_{ij}]$ is the cofactor of a_{ij} in A, and where $[A_{ij}]$ is the minor matrix obtained by deleting the ith row and jth column in A.

If

$$A = \begin{pmatrix} 2 & 1 & -1 & -3 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & 2 & -1 \\ 0 & -2 & 3 & 1 \end{pmatrix},$$

then its adjugate can be computed to be

$$A' = \begin{pmatrix} -7 & 4 & 5 & -7 \\ 7 & -14 & -13 & 11 \\ 1 & 2 & 1 & 1 \\ -20 & 14 & 16 & -14 \end{pmatrix}.$$

Now recall that, taking $m = 2$, we can obtain the 2×2 minor $\begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$ by deleting the second and third rows and columns from matrix A. Its corresponding minor in A' is $\begin{pmatrix} -7 & -7 \\ -20 & -14 \end{pmatrix}$, and its complementary minor in A is $\begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}$, which is $\text{int}A$. Thus, by Jacobi's theorem, we have

$$\det \begin{pmatrix} -7 & -7 \\ -20 & -14 \end{pmatrix} = \det A \cdot \det(\text{int } A),$$

from which we again deduce that $\det A = 6$.

Using modern terminology, we can state the general theorem as follows:

Theorem 2.[2] *Let A be an $n \times n$ matrix, let $[A_{ij}]$ be an $m \times m$ minor of A, where $m < n$, let $[A'_{ij}]$ be the corresponding $m \times m$ minor of A', and let $[A^*_{ij}]$ be the complementary $(n - m) \times (n - m)$ minor of A. Then*

$$\det[A'_{ij}] = (\det A)^{m-1} \cdot \det[A^*_{ij}].$$

Of course, what Dodgson noticed was that Jacobi's theorem provides a useful algorithm for finding $\det A$. Since this theorem is therefore central to Dodgson's method and is not universally known, we will outline a proof of it, letting the reader fill in the gaps where appropriate.

[2] In his 1833 paper, Jacobi [**10**] used the notation $\sum \pm \alpha'_1 \alpha''_2 \ldots \alpha^{(n)}_n$ to represent $\det A$ and $\sum \pm \beta'_1 \beta''_2 \ldots \beta^{(n)}_n$ to represent the adjugate of A, so he stated the general version of his theorem as $\sum \pm \beta'_1 \beta''_2 \ldots \beta^{(m)}_m = \left(\sum \pm \alpha'_1 \alpha''_2 \ldots \alpha^{(n)}_n\right)^{m-1} \sum \pm \alpha^{(m+1)}_{m+1} \alpha^{(m+2)}_{m+2} \ldots \alpha^{(n)}_n$.

Outline of proof. Without loss of generality, it is sufficient to prove Jacobi's theorem for the case of consecutive minors.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix},$$

and suppose that $[A_{ij}]$ is the $m \times m$ minor in the upper left corner of A.

By Laplace's theorem, $A \cdot A' = \det A \cdot I$, so $\det(A \cdot A') = (\det A)(\det A') = (\det A)^n$.

Now modify A' to form the $n \times n$ matrix

$$A'_I = \begin{pmatrix} a'_{11} & a'_{12} & \cdot & \cdot & \cdot & a'_{1m} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a'_{21} & a'_{22} & \cdot & \cdot & \cdot & a'_{2m} & 0 & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ a'_{m1} & a'_{m2} & \cdot & \cdot & \cdot & a'_{mm} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ a'_{(m+1)1} & a'_{(m+1)2} & \cdot & \cdot & \cdot & a'_{(m+1)m} & 1 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & & & & \cdot & 0 & 1 & & & & \cdot \\ \cdot & & \cdot & & & \cdot & 0 & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot & \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & & & & \cdot & 0 \\ a'_{n1} & a'_{n2} & \cdot & \cdot & \cdot & a'_{nm} & 0 & 0 & \cdot & \cdot & 0 & 1 \end{pmatrix}.$$

Note that we have replaced the last $n - m$ columns of A' with the corresponding columns from the $n \times n$ identity matrix, and that the $m \times m$ minor in the top left corner of A'_I is $[A'_{ij}]$. Then

$$A \cdot A'_I = \begin{pmatrix} \det A & 0 & \cdot & \cdot & 0 & a_{1(m+1)} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & \cdot & \cdot & \cdot & 0 & a_{2(m+1)} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 & \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \det A & a_{m(m+1)} & \cdot & \cdot & \cdot & a_{mn} \\ 0 & \cdot & \cdot & \cdot & 0 & a_{(m+1)(m+1)} & \cdot & \cdot & \cdot & a_{(m+1)n} \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & a_{n(m+1)} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}.$$

The determinant of this product is

$$(\det A) \cdot (\det A'_I) = \det(A \cdot A'_I) = (\det A)^m \cdot \det[A^*_{ij}].$$

And, since $\det(A'_I) = \det[A'_{ij}]$,

$$\det[A'_{ij}] = (\det A)^{m-1} \cdot \det[A^*_{ij}].$$

“De-cyphering” determinants. Since Dodgson’s algorithm restricts itself entirely to the calculation of 2×2 determinants, it is often simpler than the usual method, even in the case of large matrices. For example, in the 5×5 case, instead of carrying out 120 multiplications via the usual process, Dodgson’s method requires only 60 (plus a few divisions, of course).

Of course, its principal weakness is that it is not well-adapted to handling zeros, or *cyphers* as Dodgson called them. Since the algorithm relies on dividing by numbers in the interior of a matrix, if zeros do occur in the interior, they have to be removed by elementary row or column operations. However, this is not always possible, and in such cases, the method will break down. But despite this weakness, in many cases the algorithm can still be used effectively.

To illustrate this, we will give an example from Dodgson’s 1866 paper showing how to deal with zeros occurring during the process of condensation [**7**, p. 152]. Condensing the 5×5 matrix

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{pmatrix},$$

we get

$$A^{(1)} = \begin{pmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \end{pmatrix},$$

and condensing again gives

$$A^{(2)} = \begin{pmatrix} -30 & 6 & -12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{pmatrix}.$$

Unfortunately, the zero in the interior of this matrix makes it impossible to continue the process. However, if we use permissible elementary row operations, the determinant of the resulting matrix will be the same as the determinant of the original matrix. For our matrix, we simply need to move the top row to the bottom and move all the other rows up one. This gives us

$$A^{*} = \begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \\ 2 & -1 & 2 & 1 & -3 \end{pmatrix}.$$

We now perform the condensation method on this to obtain

$$A^{*(1)} = \begin{pmatrix} -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \\ 3 & -5 & 1 & 1 \end{pmatrix},$$

and again to get

$$\begin{pmatrix} 0 & 0 & -6 \\ 6 & 6 & -16 \\ 34 & -8 & 4 \end{pmatrix}.$$

Notice that the troublesome zeros have now moved out of the interior. Divide each term by the corresponding term in the interior of the original matrix to obtain

$$A^{*(2)} = \begin{pmatrix} 0 & 0 & 6 \\ 6 & -6 & 8 \\ -17 & 8 & -4 \end{pmatrix}.$$

Condensing this gives

$$\begin{pmatrix} 0 & 36 \\ -54 & -40 \end{pmatrix},$$

and dividing by the interior entries, we arrive at

$$A^{*(3)} = \begin{pmatrix} 0 & 12 \\ 18 & 40 \end{pmatrix}.$$

The determinant of this is -216. And finally, dividing by -6, we find that the determinant of the original matrix is 36.

Epilogue. Now that we have seen a basic overview of Dodgson's condensation method and the mathematics behind it, two simple questions remain:

1. *How or where did Dodgson come across Jacobi's theorem?*

As noted above, Dodgson was certainly not what could be described as an active research mathematician. Indeed, he did not belong to any mathematical or scientific societies, nor did he subscribe to the major mathematics research journals of the day. It is therefore very unlikely that he came across Jacobi's theorem in its original form while perusing a copy of *Crelle's Journal*. So he must have learnt about it from a secondary source. But where? There are two possibilities.

As an Oxford-bred mathematician, Dodgson would no doubt have come into contact with the very first textbook on the subject, *Elementary Theorems relating to Determinants*, published by his elder Oxford contemporary, the English mathematician William Spottiswoode, in 1851. Indeed, because of the latter's authority on the subject of determinants, Dodgson had contacted Spottiswoode with a question concerning computations in early 1866. Spottiswoode was later called upon to referee Dodgson's paper for the Royal Society [**1**, p. 331]. Given then that the two men were in contact, and that Dodgson was aware of Spottiswoode's expertise on the matter, it seems highly likely that Dodgson had read Spottiswoode's book. If so, he could not fail to have noticed a whole section devoted to proving Jacobi's theorem [**17**].

But there is no mention of Spottiswoode in any of Dodgson's work on determinants. However, in the preface to his *Elementary Treatise on Determinants*, he notes that, of the 70 propositions contained in the book, "ten are substantially taken from Baltzer's treatise on Determinants" [**8**, p. iii]. By this he is referring to a German textbook, *Theorie und Anwendung der Determinanten*, published by Richard Baltzer in 1857, which, sure enough, also contains Jacobi's theorem [**4**]. This excellent book was quickly translated into French in 1861, and reached a second edition in 1864. By 1865, Dodgson was writing in his diary: "I have been at work for some days on an elementary pamphlet on Determinants which I think of printing" [**1**, p. 334]. Could it have been Baltzer's work that prompted Dodgson's interest in the subject? Maybe. Or was it Spottiswoode's book that had first piqued his interest as a student in the 1850s and Baltzer's that reignited it ten years later? Perhaps. But like so many questions in the history of mathematics, we can never really know.

2. *Given that Dodgson's method is so efficient, justified, and useful, why is it not better known?*

For a start, Dodgson's *Treatise on Determinants* was not a bestseller, not even selling enough copies to warrant a second edition, in contrast to his other mathematical publications, many of which ran into several editions. Thus, most mathematicians would not only have been unaware of his method, but also much of the content of the book. This is illustrated by the fact that, although this work contains the first appearance in print of a well-known theorem involving simultaneous linear equations ("For solutions to exist, the rank of the augmented coefficient matrix must equal the rank of the original coefficient matrix"), this result has never been credited to him, and in fact is now known as the Kronecker-Capelli Theorem. In addition to this, even if the book had been widely read, the reader would have been slowed down and confused by the cumbersome terminology employed. Dodgson insisted on using his own rather odd names and notation instead of standard ones. For example, instead of "matrix," he used the word "block," and instead of "$a_{11}, a_{12}, a_{13}, \dots, a_{nn}$" for the matrix entries, he would write "$1 \wr 1, 1 \wr 2, 1 \wr 3, \dots, n \wr n$." Thus the limited availability of the book together with the obscurity of the text itself made it highly unlikely that Dodgson's algorithm would catch on.[3] Even Dodgson himself said that he regarded it merely as "a fanciful addition to the processes already in use" [**8**, p. v]. Nevertheless, when teaching linear algebra, we have consistently found Dodgson's method to be the most popular method among our students for evaluating large determinants. Curious!

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[3]Despite its widespread absence from most linear algebra textbooks, a book by Turnbull from 1960 does contain a very small section on "Dodgson's Method of Evaluating a Determinant by Condensation" [**18**, p. 340].

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12. C. Jacobi, De formatione et proprietatibus Determinantium, *Journal für die reine und angewandte Mathematik* **22** (1841) 285–318.
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18. H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, 3rd ed., Dover, 1960.
19. R. J. Wilson, Alice in Numberland: An Informal Dramatic Presentation, *College Math. J.* **33** (2002) 354–377.

Ted Ridgway (ridgwat@arc.losrios.cc.ca.us), American River College, detected the same—unfortunately not uncommon—error involving the use of the word "percent" in two books that he read recently.

From *A Short History of Nearly Everything* by Bill Bryson (Broadway Books, 2003), page 16:

> Rees maintains that six numbers in particular govern our universe, and that if any of these values were changed even very slightly things could not be as they are. For example, for the universe to exist as it does requires that hydrogen be converted to helium in a precise but comparatively stately manner—specifically, in a way that converts seven one-thousandths of its mass to energy. Lower that value very slightly—from 0.007 percent to 0.006 percent, say—and no transformation could take place: the universe would consist of hydrogen and nothing else. Raise the value very slightly—to 0.008 percent—and bonding would be so wildly prolific that the hydrogen would long since have been exhausted. In either case, with the slightest tweaking of the numbers the universe as we know and need it would not be here.

From *Fast Food Nation: The Dark Side of the All-American Meal* by Eric Schlosser (Houghton Mifflin, 2001), page 124 of the Perennial 2002 edition:

> The human nose, however, is still more sensitive than any machine yet invented. A nose can detect aromas present in quantities of a few parts per trillion—an amount equivalent to 0.000000000003 percent.