Directions: Please answer all questions in the space provided. Use of calculators or any form of electronic communication device is strictly forbidden on this quiz.

1. A matrix A is symmetric if $A^T = A$. Find a basis for the vector space of all $3 \times 3$ symmetric matrices.

If $A$ is a $3 \times 3$ symmetric matrix, then the condition $A^T = A$ means $A$ has form $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, for $a, b, c, d, e, f \in \mathbb{R}$.

Let $V = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}$ be the set of all symmetric $3 \times 3$ matrices (which is easily seen to be a subspace of $M_{3,3}$). Notice that for any symmetric matrix $A$ we have

$$A = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $V$. Also, the set $S$ is linearly independent because if

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and consequently $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$.

So we have seen that the set $S$ spans $V$ and is linearly independent. Conclusion:

The set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $V$.

You had one of the following problems for your second question:

2. Consider the following subspace of $\mathbb{R}^4$: $W = \{(t, 3s - 4t, t + s, s) : s$ and $t$ are real numbers$\}$

Find a basis for $W$. State the dimension of $W$.

Note that $(t, 3s - 4t, t + s, s) = s(0, 3, 1, 1) + t(1, -4, 1, 0)$.
This means any vector in $W$ is a linear combination of $(0, 3, 1, 1)$ and $(1, -4, 1, 0)$.
Thus the set $S = \{(0, 3, 1, 1), (1, -4, 1, 0)\}$ spans $W$.
Moreover, the set $S$ of two vectors is linearly independent because no vector in $S$ is a multiple of the other.
Since $S$ is linearly independent and spans $W$, it is a basis for $W$.
Since the basis has two elements, the subspace $W$ has dimension 2.
Conclusion: The set $\{(0,3,1,1), (1,-4,1,0)\}$ is a basis for $W$, and $W$ is two-dimensional.

3. Consider the following subspace of $\mathbb{R}^4$: $W = \{(2s - 3t, s, t + s, t) : s$ and $t$ are real numbers$\}$

Find a basis for $W$. State the dimension of $W$.

Note that $(2s - 3t, s, t + s, t) = s(2, 1, 1, 0) + t(-3, 0, 1, 1)$.
This means any vector in $W$ is a linear combination of $(2, 1, 1, 0)$ and $(-3, 0, 1, 1)$.
Thus the set $S = \{(2, 1, 1, 0), (-3, 0, 1, 1)\}$ spans $W$.
Moreover, the set $S$ of two vectors is linearly independent because no vector in $S$ is a multiple of the other.
Since $S$ is linearly independent and spans $W$, it is a basis for $W$.
Since the basis has two elements, the subspace $W$ has dimension 2.
Conclusion: The set $\{(2,1,1,0), (-3,0,1,1)\}$ is a basis for $W$, and $W$ is two-dimensional.