1. Suppose \(u, v, w\) are three vectors in a vector space \(V\). Without knowing any further information, is it possible to say whether or not the set \(\{v - u, w - v, u - w\}\) is linearly independent or dependent?

Notice that \(1(v - u) + 1(w - v) + 1(u - w) = 0\), so it follows the set \(S\) is **linearly dependent**.

Note: One common mistake was to take the equation \((c_3 - c_1)u + (c_1 - c_2)v + (c_2 - c_3)w = 0\), get \(c_3 - c_1 = 0, c_1 - c_2 = 0, c_2 - c_3 = 0\), and solve the resulting system.

But there is a problem with this approach. Since we are not given that the vectors \(u, v, w\) are linearly independent, we can’t deduce that \(c_3 - c_1 = 0, c_1 - c_2 = 0, c_2 - c_3 = 0\).

2. Does the set \(S = \{1 + x, x + x^2, x^2 + x^3, 1 + x^3\}\) span \(P_3\)?

Given an arbitrary polynomial \(a + bx + cx^2 + dx^3\), we want to know if we can always find values for \(c_1, c_2, c_3, c_4\) satisfying \(c_1(1 + x) + c_2(x + x^2) + c_3(x^2 + x^3) + c_4(1 + x^3) = a + bx + cx^2 + dx^3\).

Combining, we get \((c_1 + c_4) + (c_1 + c_2)x + (c_2 + c_3)x^2 + (c_3 + c_4)x^3 = a + bx + cx^2 + dx^3\), which leads to the following system.

\[
\begin{align*}
1 &+ c_4 = a \\
c_1 + c_2 &= b \\
c_2 + c_3 &= c \\
c_3 + c_4 &= d
\end{align*}
\]

Solving:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & a \\
1 & 1 & 0 & b \\
0 & 1 & 1 & c \\
0 & 0 & 1 & 1 & d
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 1 & a \\
0 & 1 & 0 & -1 & b - a \\
0 & 1 & 1 & 0 & c \\
0 & 0 & 1 & 1 & d
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 1 & a \\
0 & 1 & 0 & -1 & b - a \\
0 & 0 & 1 & 1 & c - b + a \\
0 & 0 & 0 & 1 & d - c + b - a
\end{pmatrix}
\]

Looking at the last row, you can see the system has no solutions for some values of \(a, b, c\) and \(d\). In particular if \(a = 0, b = 0, c = 0\) and \(d = 1\), there will be no solution.

Thus the polynomial \(0 + 0x + 0x^2 + x^3\) cannot be written as a linear combination of elements in \(S\), so \(\text{S does not span } P_3\).