☞ Quiz: Sections 4.4, 4.5 | Linear Algebra MATH 310 R. Hammack | November 1, 2018

Name: Richard | Score: 10

There are TWO questions (on front and back).

---

1. Find all real numbers t for which the set $S = \{(t,1,1),\ (1,t,1),\ (1,1,t)\}$ is linearly independent.

If t is such a number, then the equation

$$x(t,1,1) + y(1,t,1) + z(1,1,t) = (0,0,0)$$

has only the trivial solution $x=0,\ y=0,\ z=0$.
So we will solve this equation for x, y, z and see for which t we get only the trivial solution.
The above equation yields

$$(xt+y+z,\ x+yt+z,\ x+y+zt) = (0,0,0)$$

$$\Rightarrow \begin{cases} xt+y+z = 0 \\ x+yt+z = 0 \\ x+y+zt = 0 \end{cases} \Rightarrow \begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we have an equation $A\vec{x} = \vec{0}$, where A is 3x3 matrix. We have a theorem that tells us that $A\vec{x} = \vec{0}$ has only the trivial solution, provided that $|A| \neq 0$. Thus let's look at $|A|$.

$$\begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = t\begin{vmatrix} t & 1 \\ 1 & t \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & t \end{vmatrix} + 1\begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix} = t(t^2-1) - (t-1) + (1-t)$$

$$= t(t+1)(t-1) - 2(t-1) = (t-1)\big(t(t+1)-2\big) = (t-1)(t^2+t-2)$$

$$= (t-1)(t+2)(t-1) = 0$$

So the determinant is zero precisely for $t=1$, $t=-2$,

So $\boxed{S \text{ is linearly independent provided } t \neq 1 \text{ and } t = -2}$

2. Decide if the set $S = \{ (1,0,0,1), (0,2,0,2)), (1,0,1,0), (0,2,2,0) \}$ is a basis for $\mathbb{R}^4$. Explain your reasoning.

First let's check to see if S is linearly independent.

Consider $w(1,0,0,1) + x(0,2,0,2) + y(1,0,1,0) + z(0,2,2,0) = (0,0,0,0)$

We need to see whether or not there are non-trivial solutions. The above equation gives

$$(w+y,\ 2x+2z,\ y+2z,\ w+2x) = (0,0,0,0)$$

$$\Rightarrow \begin{cases} w \quad + y \quad = 0 \\ 2x \quad +2z = 0 \\ y + 2z = 0 \\ w + 2x \quad = 0 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 - R_1 \to R_4]{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_4 - 2R_2 \to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} w = 2z \\ x = -z \\ y = -2z \\ z = \underline{\text{free}} \end{array}$$

Since there is a free variable, we will have non-trivial solutions, so <u>S is not linearly independent</u>

~~Consequently:~~ $\boxed{S \text{ is } \underline{\text{not}} \text{ a basis for } \mathbb{R}^4}$