Midterm

Name: Richard

Linear Algebra
MATH 310
R. Hammack

October 16, 2018

Score: 100

Answer in the space provided. No calculators. Please put all phones, etc., away. Each problem is 10 points.

1. For this problem, $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 1 \end{bmatrix}$.

   Preform the indicated operations or state that they are not possible.

   (a) $A^T C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}}$

   (b) $DB^2 = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 2 \end{bmatrix}}$

   (c) $B^{-1} = \dfrac{1}{(2)(-2)-(2)(-1)}\begin{bmatrix} -2 & 1 \\ -2 & 2 \end{bmatrix} = -\dfrac{1}{2}\begin{bmatrix} -2 & 1 \\ -2 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -1/2 \\ 1 & -1 \end{bmatrix}}$

   (d) $|B| = (2)(-2)-(2)(-1) = \boxed{-2}$

   (e) $|B^5| = |B|^5 = (-2)^5 = \boxed{-32}$

2. Suppose X and Y are matrices for which the product XY is defined, and $c \in \mathbb{R}$.

   (a) If $XY = O$, is necessarily true that $X = O$ or $Y = O$? Justify your answer.

   No Maybe $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$ and $Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$.
   Then $XY = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

   (b) If $cY = O$, is necessarily true that $c = 0$ or $Y = O$?

   Yes

3. Solve the system $\begin{cases} 2w - x + 8y - 4z = 4 \\ w + x + 4y - 2z = 5 \\ w - 2x + 4y - 2z = -1 \end{cases}$

$$\left[\begin{array}{cccc|c} 2 & -1 & 8 & -4 & 4 \\ 1 & 1 & 4 & -2 & 5 \\ 1 & -2 & 4 & -2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -2 & 5 \\ 2 & -1 & 8 & -4 & 4 \\ 1 & -2 & 4 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_1]{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -2 & 5 \\ 0 & -3 & 0 & 0 & -6 \\ 0 & -3 & 0 & 0 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -2 & 5 \\ 0 & -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -2 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -2 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} w + 4y - 2z = 3 \\ x = 2 \end{cases}$$

$$\begin{cases} w = 3 - 4y + 2z \\ x = 2 \end{cases}$$

y & z are free

Solutions: $\boxed{(3-4y+2z,\ 2,\ y,\ z) \text{ where } y, z \in \mathbb{R}}$

4. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 4 & -1 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -1 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

Thus $\boxed{A^{-1} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}$

Check $AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

5. Suppose that $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Use your answer from problem 4 above to find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}}$$

Check!

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

✓

6. Consider the two vectors $\mathbf{u}_1 = (2, 1, 2)$ and $\mathbf{u}_2 = (4, -1, 2)$ in $\mathbb{R}^3$.
Is the vector $\mathbf{v} = (2, 7, 6)$ a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$? Explain.

Let's see. The problem is asking if there are numbers $x, y \in \mathbb{R}$ for which

$$x\vec{u}_1 + y\vec{u}_2 = \vec{v}$$

i.e. $x(2,1,2) + y(4,-1,2) = (2,7,6)$

or $(2x+4y,\ x-y,\ 2x+2y) = (2,7,6)$

To see if there are such x and y we must solve this system:

$$\begin{cases} 2x+4y = 2 \\ x-y = 7 \\ 2x+2y = 6 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 2 \\ 1 & -1 & 7 \\ 2 & 2 & 6 \end{array}\right] \xrightarrow[\frac{1}{2}R_3 \to R_3]{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 7 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array}\right] \xrightarrow[R_3 + R_2 \to R_3]{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \quad \boxed{\begin{array}{l} x = 5 \\ y = -2 \end{array}}$$

Answer $\boxed{\text{Yes, in fact } 5\vec{u}_1 - 2\vec{u}_2 = \vec{v}}$

Check: $5(2,1,2) - 2(4,-1,2) \stackrel{?}{=} (2,7,6)$

$(10,5,10) - (8,-2,4) = (2,7,6)$ ✓

7. Find the value(s) of x for which the matrix $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 4 & 2 & x \end{bmatrix}$ is not invertible.

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 0 \\ 4 & 2 & x \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 2 & x \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ 4 & x \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}$$

$$= -x - 0 + 3(4+4)$$

$$= -x + 24$$

Notice that the determinant is zero exactly when $x = 24$.

Thus $\boxed{\text{matrix is not invertible when } x = 24}$

8. Find the 2×2 matrix A for which $A \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$$A \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$A\, I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \frac{1}{3 \cdot 4 - 1 \cdot 2} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A = \frac{1}{10} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A = \frac{1}{10} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \boxed{\frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}$$

Check: $\frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \checkmark$$

9. An $n \times n$ (square) matrix A is said to be **skew-symmetric** if $A^T = -A$. Suppose n is odd and A is skew-symmetric. Find $\det(A)$.

Then
$$\det(A^T) = \det(-A)$$
$$\det(A) = \det((-1)A)$$
$$\det(A) = (-1)^n \det(A)$$
$$\det(A) = -\det(A)$$

$$2\det(A) = 0$$

$$\boxed{\det(A) = 0}$$

10. In this problem we regard $\mathbb{R}^n$ as a set of column vectors, that is, $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$.

Let A be fixed $n \times n$ matrix. Show that the set $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = -\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.

① First let's show W is closed under $+$. Let $\vec{x}, \vec{y} \in W$. This means $A\vec{x} = -\vec{x}$ and $A\vec{y} = -\vec{y}$. Notice That $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ $= -\vec{x} - \vec{y} = -(\vec{x} + \vec{y})$. That is, $A(\vec{x} + \vec{y}) = -(\vec{x} + \vec{y})$ which means $\vec{x} + \vec{y} \in W$. Thus W is closed under $+$.

② Next we show W is closed under scalar mult. Take $c \in \mathbb{R}$ and $\vec{x} \in W$. We need to show that $c\vec{x} \in W$. Now, $A\vec{x} = -\vec{x}$ because $\vec{x} \in W$. Notice that $A(c\vec{x}) = cA\vec{x} = c(-\vec{x}) = -c\vec{x}$. Thus we have $A(c\vec{x}) = -c\vec{x}$, which means $c\vec{x} \in W$. Thus W is closed under scalar mult.

Consequently W is a subspace of $\mathbb{R}^n$