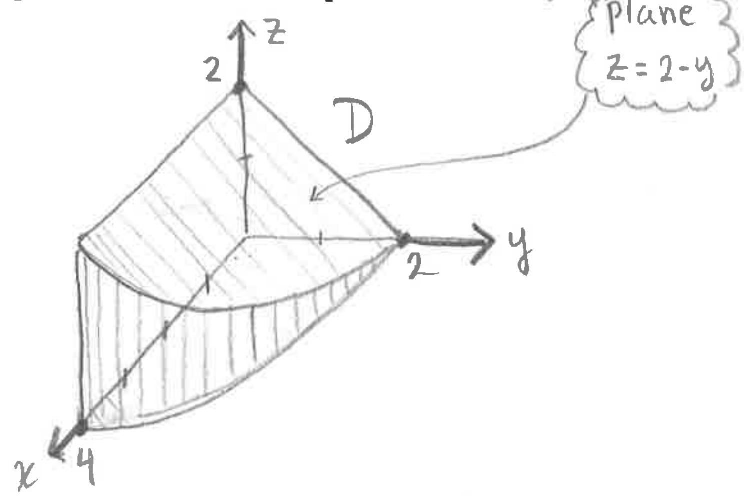
4. (25 pts.) Find the volume of the 3-D region D in the first octant, bounded by the coordinate planes, the graph of $x = 4 - y^2$, and the plane $y + z = 2$.



$$V = \iiint_D dV = \int_0^2 \int_0^{4-y^2} \int_0^{2-y} dz\, dx\, dy$$

$$= \int_0^2 \int_0^{4-y^2} \left[z\right]_0^{2-y} dx\, dy$$

$$= \int_0^2 \int_0^{4-y^2} 2 - y \; dx\, dy$$

$$= \int_0^2 \left[2x - yx\right]_0^{4-y^2} dy$$

$$= \int_0^2 2(4-y^2) - y(4-y^2)\, dy$$

$$= \int_0^2 \left(8 - 2y^2 - 4y + y^3\right) dy$$

$$= \left[8y - \frac{2}{3}y^3 - 2y^2 + \frac{y^4}{4}\right]_0^2$$

Good Luck!

$$= 8 \cdot 2 - \frac{2}{3} \cdot 8 - 2 \cdot 4 + \frac{16}{4}$$

$$= 16 - \frac{16}{3} - 8 + 4 = 12 - \frac{16}{3}$$

$$= \frac{36}{3} - \frac{16}{3} = \boxed{20/3 \text{ cubic units}}$$

VCU

MATH 307
Multivariate Calculus

R. Hammack

Sample Test 3

November 5, 2013

Name: Richard

Score: 100

Directions. Solve the following questions in the space provided. Unless noted otherwise, you must show your work to receive full credit. This is a closed-book, closed-notes test. Calculators, computers, etc., are not used. Put a your final answer in a box, where appropriate.

1. (25 points) Consider the integral $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$.

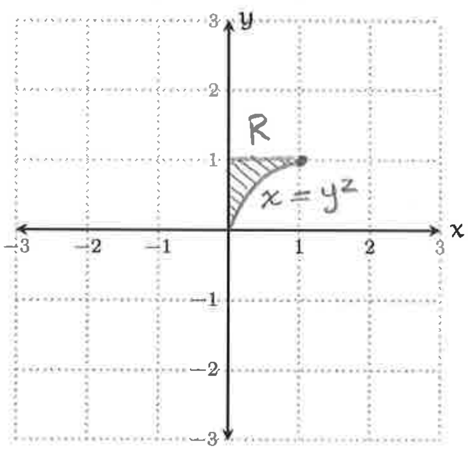
(a) Evaluate the integral.

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy = \int_0^1 \left[3y^3 \frac{1}{y} e^{xy} \right]_0^{y^2} dy = \int_0^1 \left[3y^2 e^{xy} \right]_0^{y^2} dy$$

$$= \int_0^1 \left(3y^2 e^{y^2 y} - 3y^2 e^{0y} \right) dx = \int_0^1 \left(e^{y^3} 3y^2 - 3y^2 \right) dy$$

$$= \left[e^{y^3} - y^3 \right]_0^1 = \left(e^{1^3} - 1^3 \right) - \left(e^{0^3} - 0^3 \right) = e - 1 - 1 = \boxed{e - 2}$$

(b) Sketch the region of integration.



$x = y^2$ is same curve as $y = \sqrt{x}$

(c) Write an equivalent double integral with the order of integration reversed. (You do not need to evaluate it.)

$$\boxed{\int_0^1 \int_{\sqrt{x}}^1 3y^3 e^{xy}\, dy\, dx}$$

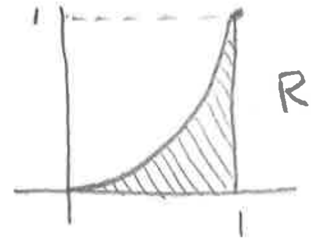
(c) Find the average value of the function $f(x, y) = 3y^3 e^{xy}$ on the region sketched in part (b) above.

$$\text{Ave. value} = \frac{\iint_R 3y^3 e^{xy}\, dA}{\text{Area of } R} = \frac{e-2}{\frac{1}{3}} = \boxed{3e - 6}$$

$$\text{Area of } R = \int_0^1 y^2\, dy = \left[\frac{y^3}{3} \right]_0^1 = \frac{1}{3}$$

2. (25 pts.) Find the center of mass of the region in the first quadrant of the plane that is bounded by the curve $y = x^2$, the x-axis, and the line $x = 1$. (Assume a constant density of $\delta(x, y) = 1$.)

$$\text{Mass} = \iint_R dA = \int_0^1 \int_0^{x^2} dy\,dx$$



$$= \int_0^1 [y]_0^{x^2} dx = \int_0^1 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$M_x = \iint_R x\,dA = \int_0^1 \int_0^{x^2} x\,dy\,dx = \int_0^1 [xy]_0^{x^2} dx$$

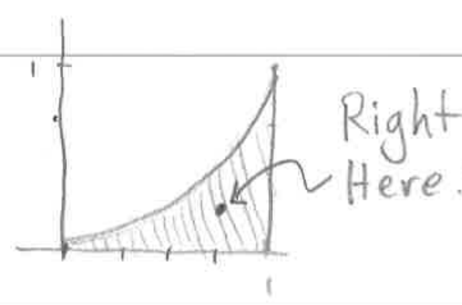
$$= \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$M_y = \iint_R y\,dA = \int_0^1 \int_0^{x^2} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{x^2} dx$$

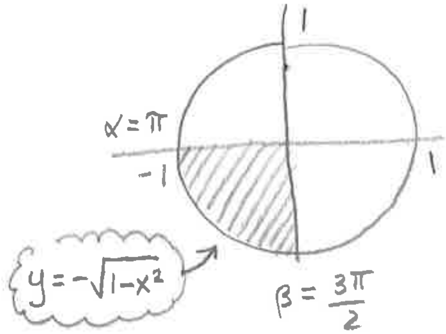
$$= \int_0^1 \frac{x^4}{2} dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10}$$

Center of mass:

$$(\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{\frac{1}{4}}{\frac{1}{3}}, \frac{\frac{1}{10}}{1/3}\right) = \boxed{\left(\frac{3}{4}, \frac{3}{10}\right)}$$



3. (25 pts.) Evaluate $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$ by switching to polar coordinates and evaluating the resulting integral.



The region of integration is a quarter circle, of radius 1, as shown. It lies between the polar angles $\alpha = \pi$ and $\beta = \frac{3\pi}{2}$. The above integral becomes

$$\int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \frac{2}{1+\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}\, r\,dr\,d\theta$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 \frac{2r}{1+r}\,dr\,d\theta$$

$$\begin{array}{r} 2 \\ 1+r\,\overline{)\,2r} \\ \underline{2r+2} \\ -2 \end{array}$$

$$\therefore \frac{2r}{1+r} = 2 - \frac{2}{1+r}$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 2 - \frac{2}{1+r}\,dr\,d\theta$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \Big[2r - 2\ln|1+r|\Big]_0^1\,d\theta$$

$$= \int_{\pi}^{3\pi/2} \big(2\cdot 1 - 2\ln|1+1|\big) - \big(2\cdot 0 - 2\ln|1+0|\big)\,d\theta$$

$$= \int_{\pi}^{3\pi/2} 2 - 2\ln 2\,d\theta = (2 - 2\ln 2)\int_{\pi}^{\frac{3\pi}{2}} d\theta$$

$$= (2 - \ln 2^2)\big[\theta\big]_{\pi}^{\frac{3\pi}{2}} = (2 - \ln 4)\frac{\pi}{2}$$

$$= \pi\left(1 - \tfrac{1}{2}\ln 4\right) = \pi\left(1 - \ln\sqrt{4}\right) = \boxed{\pi(1 - \ln 2)}$$