1. (30 pts.) Consider function $z = f(x, y) = \ln(x^2 + y^2)$.

(a) State the domain of $f$. All points $(x, y)$ in the plane except $(0,0)$.

(b) State the range of $f$. All real numbers.

(c) $f(0, \frac{1}{e}) = \ln\left(0^2 + \left(\frac{1}{e}\right)^2\right) = \ln\left(\frac{1}{e^2}\right) = \boxed{-2}$

(d) Sketch the level curve for $z = \ln(4)$.

\[ \ln(4) = \ln(x^2 + y^2) \]
\[ 4 = x^2 + y^2 \]
\[ x^2 + y^2 = 4 \]
\[ x^2 + y^2 = r^2 \]

(e) \[ \nabla f(x, y) = \begin{pmatrix} \frac{2x}{x^2 + y^2} \\ \frac{2y}{x^2 + y^2} \end{pmatrix} \]

(f) Find the rate of change of $f(x, y)$ in the direction of $(5, 5)$ at the point $(1, 3)$.

Direction is $\vec{u} = \langle 5, 5 \rangle = \frac{\langle 5, 5 \rangle}{\sqrt{50}} = \frac{5\sqrt{2}}{5} = \langle \frac{\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}} \rangle$.

Rate of change at $(x, y)$ is $\nabla f(x, y) = \nabla f \cdot \vec{u} = \langle \frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

Rate of change at $(1, 3)$ is thus $\langle \frac{3}{10}, \frac{2}{10} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{8}{10\sqrt{2}} = \frac{4}{5\sqrt{2}} = \boxed{\frac{2\sqrt{2}}{5}}$. 

2. (24 pts.) Evaluate each limit, if possible; if not, explain why it does not exist.

(a) \[ \lim_{(x,y) \to (0,0)} \frac{x-y}{x+y} \]
Consider the following two approaches \((x,y) \to (0,0)\):

- Along the x-axis (\(y=0\)): \[ \lim_{(x,y) \to (0,0)} \frac{x-y}{x+y} = \lim_{(x,y) \to (0,0)} \frac{x}{x} = 1 \]
- Along the y-axis (\(x=0\)): \[ \lim_{(x,y) \to (0,0)} \frac{x-y}{x+y} = \lim_{(x,y) \to (0,0)} \frac{-y}{y} = -1 \]

Since we get different values along different paths, limit \[ \text{DNE} \]

(b) \[ \lim_{(x,y) \to (1,1)} \frac{xy - y - 2x + 2}{x-1} = \lim_{(x,y) \to (1,1)} \frac{y(x-1) - 2(x-1)}{x-1} = \lim_{(x,y) \to (1,1)} \frac{(x-1)(y-2)}{x-1} = \lim_{(x,y) \to (1,1)} (y-2) = -1 \]

3. (20 pts.) Consider the function \(f(x,y) = e^{4x-x^2-y^2} \). Find all local maxima, local minima and/or saddle points.

\[ \nabla f(x,y) = \langle e^{4x-x^2-y^2}(y-2x), -e^{4x-x^2-y^2}(y+2y) \rangle = \langle 0, 0 \rangle \]

From this we see that there is one critical point \((2,0)\)

\[ f_{xx}(x,y) = \frac{4x}{e^{4x-x^2-y^2}}(y-2x)^2 + \frac{4x}{e^{4x-x^2-y^2}}(-2) \]
\[ f_{xx}(2,0) = e^4(4-2)^2 + e^4(-2) = -2e^4 \]

\[ f_{yy}(x,y) = \frac{4y}{e^{4x-x^2-y^2}}y^2 - \frac{4y}{e^{4x-x^2-y^2}}y^2 = 0 \]
\[ f_{yy}(2,0) = e^4(0) + 2e^4 = -2e^4 \]

\[ f_{xy}(x,y) = e^{4x-x^2-y^2}(-2y)(4-2x) \]
\[ f_{xy}(2,0) = 0 \]

Now, \[ f_{xx}(2,0)f_{yy}(2,0) - f_{xy}(2,0)^2 = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0 \]
Also \[ f_{xx}(2,0) = -2e^4 < 0 \]

Therefore there is a local maximum at \((2,0)\)
4. (16 pts.) Consider \( f(x, y) = \ln(xy) \tan^{-1}(x) \).

(a) \( \frac{\partial f}{\partial x} = \frac{y}{xy} \tan^{-1}(x) + \ln(xy) \frac{1}{1 + x^2} = \frac{\tan^{-1}(x)}{x} + \frac{\ln(xy)}{1 + x^2} \)

(b) \( \frac{\partial f}{\partial y} = \frac{x}{xy} \tan^{-1}(x) = \frac{\tan^{-1}(x)}{y} \)

(c) \( \frac{\partial^2 f}{\partial y \partial x} = \frac{x}{xy} \frac{1}{1 + x^2} = \frac{1}{y(1 + x^2)} \)

(d) \( f_x(1, 1) = \frac{\tan^{-1}(1)}{1} + \frac{\ln(1.1)}{1 + 1^2} = \frac{\pi}{4} + \frac{0}{2} = \frac{\pi}{4} \)

5. (10 pts.) Sketch the domain of

\[ f(x, y) = \frac{\sqrt{1 - x + y}}{x + 2} \]

Need \( 1 - x + y \geq 0 \) \( \Rightarrow \{ y \geq x - 1 \} \)

and \( x + 2 \neq 0 \) \( \Rightarrow x \neq -2 \)

(line \( x = -2 \) not included)