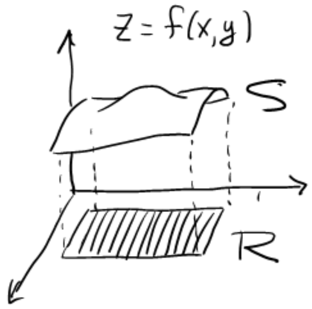
Section 16.6 Surface Integrals

Recall: If a surface S is parameterized as $\vec{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle$ where the (u,v) are from a region R on the uv-plane. Then area of S is

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \underbrace{\int_a^b \int_c^d |r_u \times r_v| \, du \, dv}_{\text{(Provided } R \text{ is rectangle } a \le u \le b,\ c \le v \le d.)}$$

<u>Remark</u> Suppose a surface is defined explicitly as the graph of $z = f(x,y)$ for $a \le x \le b$, $c \le x \le d$. Then it is automatically parameterized as $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$.

Then $\vec{r}_x = \langle 1, 0, f_x \rangle \quad \vec{r}_y = \langle 0, 1, f_y \rangle$



So $\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$

and $|\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1}$.

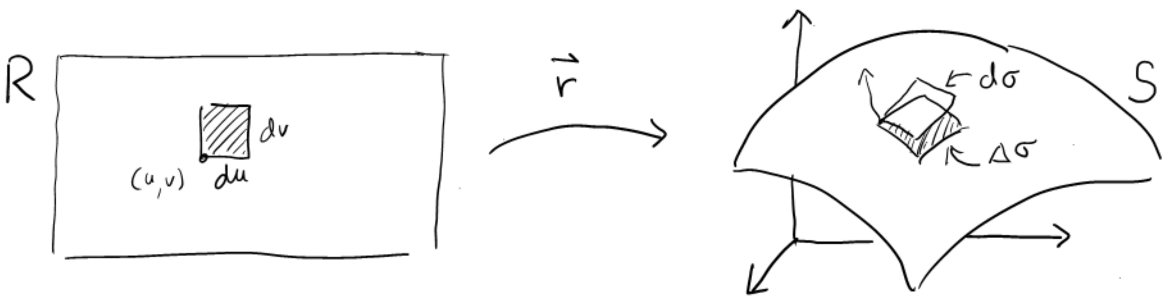
Therefore its area is

Provided R is the rectangle $a \le x \le b$, $c \le y \le d$

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dA = \int_a^b \int_c^d \underbrace{\sqrt{f_x^2 + f_y^2 + 1}}_{|\vec{r}_x \times \vec{r}_y|} \, dx \, dy$$

Now, since explictly defined surfaces are also parametric surfaces, we focus on parametric

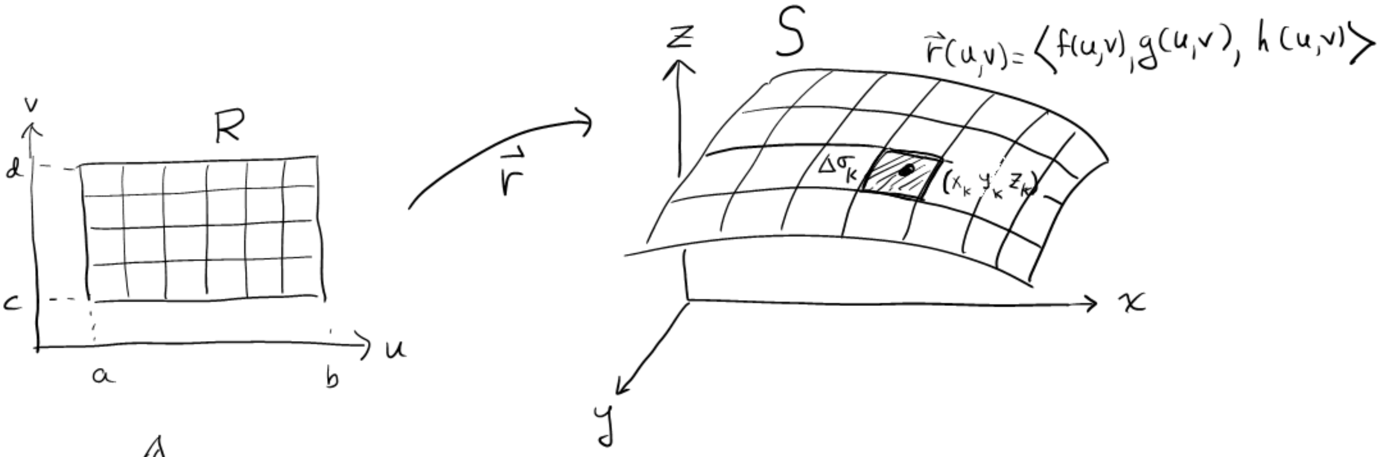
<u>Definition</u> The <u>area</u> <u>differential</u> is $d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$



At (u,v) a rectangle of dimensions $du \cdot dv$ is mapped via $\vec{r}$ to a curved rectangle of area $\Delta\sigma$. It turn, $\Delta\sigma$ is approximated by $d\sigma = \underline{|\vec{r}_u \times \vec{r}_v| \, du \, dv}$

We sometimes write $A = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \iint_S d\sigma$

Surface Integrals



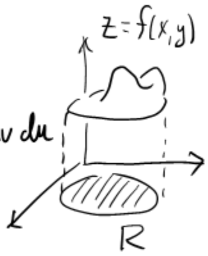
Note R need not be a rectangle, but it *is* in most of the text's examples

Suppose a function $G(x, y, z)$ is defined on the surface S. Divide S into n curved rectangles of areas $\Delta\sigma_1 \; \Delta\sigma_2 \; \Delta\sigma_3 \cdots \Delta\sigma_n$. In the k^{th} rectangle put a sample point (x_k, y_k, z_k). The <u>surface integral of G</u> over S is defined to be

$$\iint_S G(x,y,z)\,d\sigma = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k, y_k, z_k)\,\Delta\sigma_k$$

In the above setting, $\displaystyle\iint_S G(x,y,z)\,d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dv\,du$

If S is $z = f(x,y)$ then $\displaystyle\iint_S G(x,y,z)\,d\sigma = \iint_R G(x, y, f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\;dv\,du$



Surface integrals have various interpretations and meanings

- For example if the surface is a sheet of metal with electrical charge $G(x,y,z)$ at point (x,y,z), then the total charge is

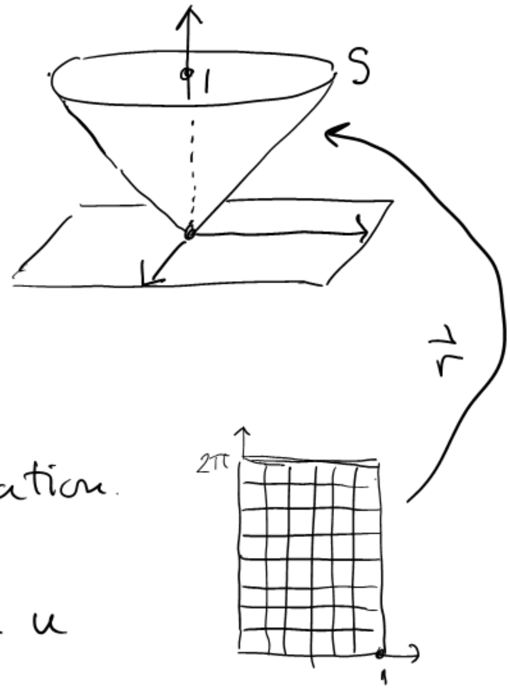
$$\iint_S G(x,y,z)\,d\sigma$$

If $G(x,y,z)$ is density at (x,y,z) the integral gives mass.

- Also, given v.f. F and surface S, the flux across S is given by

$$\text{Flux} = \iint_S F \cdot \vec{n}\,d\sigma$$

But our examples will concentrate on how to compute surface integrals.



Example

Let $G(x,y,z) = z - x$, and suppose

S is cone $\quad z = \sqrt{x^2+y^2}$, $0 \le z \le 1$

Compute $\displaystyle\iint_S G(x,y,z)\, d\sigma$

Solution First we find a parameterization.

Let $x = u\cos v$, $\quad y = u \sin v$.

Then $z = \sqrt{x^2+y^2} = \sqrt{(u\cos v)^2 + (u \sin v)^2} = u$

Therefore S is given by $\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle$, $0 \le u \le 1$, $0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v,\ \sin v,\ 1\rangle$

$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -u\cos v,\ u \sin v,\ u\cos^2 v + u\sin^2 u\rangle = \langle -u\cos v,\ u\sin v,\ u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\sin^2 v + u^2\cos^2 v + u^2} = \sqrt{u^2 + u^2} = \sqrt{2u^2} = u\sqrt{2}$$

$$\iint_S G(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^1 G(u\cos v,\ u\sin v,\ u)\,|\vec{r}_u \times r_v|\, du\, dv$$

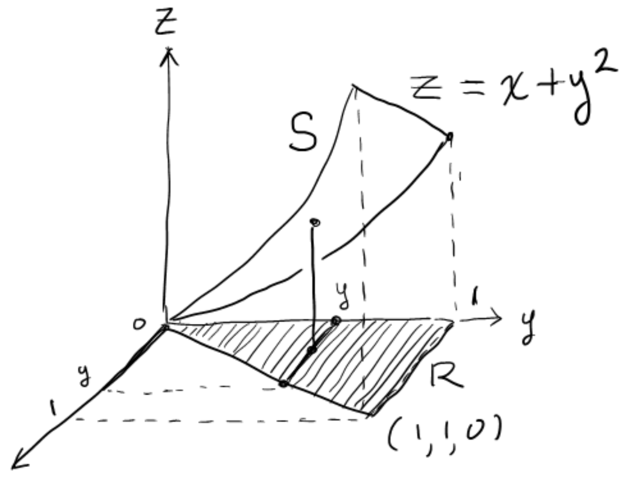
$$= \int_0^{2\pi}\int_0^1 (u - u\cos v)\, u\sqrt{2}\, du\, dv = \int_0^{2\pi}\int_0^1 \sqrt{2}u^2(1-\cos v)\, du\, dv$$

$$= \int_0^{2\pi}\left[\frac{\sqrt{2}}{3}u^3(1-\cos v)\right]_0^1 dv = \frac{\sqrt{2}}{3}\int_0^{2\pi}(1-\cos v)\, dv$$

$$= \frac{\sqrt{2}}{3}\Big[v - \sin v\Big]_0^{2\pi} = \boxed{\frac{2\sqrt{2}\,\pi}{3}}$$

Example

16.6 (15) Suppose $G(x,y,z) = z - x$ and S is as indicated.



$$\iint_S G(x,y,z)\, d\sigma$$

$$= \iint_R G(x, y, f(x,y)) \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$= \int_0^1 \int_0^y G(x, y, x+y^2) \sqrt{1^2 + (2y)^2 + 1}\, dx\, dy$$

$$= \int_0^1 \int_0^y (x + y^2 - x) \sqrt{2 + 4y^2}\, dx\, dy$$

$$= \int_0^1 \int_0^y y^2 \sqrt{2+4y^2}\, dx\, dy = \int_0^1 \left[x y^2 \sqrt{2+4y^2} \right]_0^y dx\, dy$$

$$= \int_0^1 y^3 \sqrt{2 + 4y^2}\, dy = \left[\frac{y^2}{12} \sqrt{2+4y^2}^{\,3} - \frac{1}{120} \sqrt{2+4y^2}^{\,5} \right]_0^1 = \left(\frac{\sqrt{6}^{\,3}}{12} - \frac{\sqrt{6}^{\,5}}{120} \right) - \left(\frac{\sqrt{2}^{\,3}}{12} - \frac{\sqrt{2}^{\,5}}{120} \right)$$

$$= \frac{6\sqrt{6}}{12} - \frac{36\sqrt{6}}{120} - \frac{2\sqrt{2}}{12} + \frac{4\sqrt{2}}{120} = \boxed{\frac{\sqrt{6}}{5} - \frac{2\sqrt{2}}{15}}$$

Integration By Parts:

$$\int y^3 \sqrt{2+4y^2}\, dy = \int y^2 \sqrt{2+4y^2}\, y\, dy = uv - \int v\, du$$

$$= \frac{y^2}{12} \sqrt{2+4y^2}^{\,3} - \int \frac{1}{12} \sqrt{2+4y^2}^{\,3}\, 2y\, dy$$

$$= \frac{y^2}{12} \sqrt{2+4y^2}^{\,3} - \frac{1}{48} \int \sqrt{2+4y^2}^{\,3}\, 8y\, dy$$

$$= \frac{y^2}{12} \sqrt{2+4y^2}^{\,3} - \frac{1}{120} \sqrt{2+4y^2}^{\,5}$$

$u = y^2$ $\quad dV = \sqrt{2+4y^2}\, y\, dy$

$du = 2y\, dy$ $\quad V = \frac{1}{8} \int \sqrt{2+4y^2}\, 8y\, dy$

$$= \frac{1}{12} \sqrt{1+2y^2}^{\,3}$$