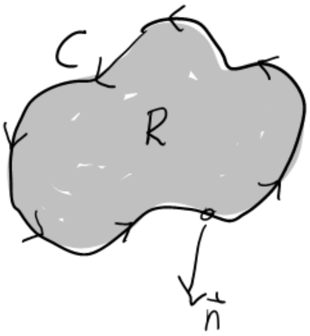
Section 16.4 Green's Theorem

Green's Theorem states the following:

> Suppose $\vec{r}(t)$ is a closed curve enclosing a region R on the plane, and $F = \langle M(x,y), N(x,y)\rangle = \langle M, N\rangle$ is a vector field. Then:
>
> $$\oint_C F\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dy\,dx$$

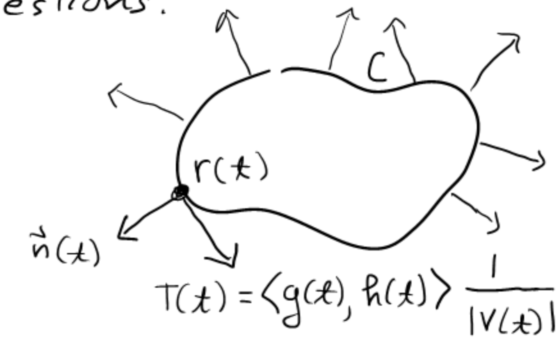


What does this mean? Why is it true? How is it useful? Today we will seek answers to these questions.

Topic 1 The unit normal $\vec{n}(t)$ to C is

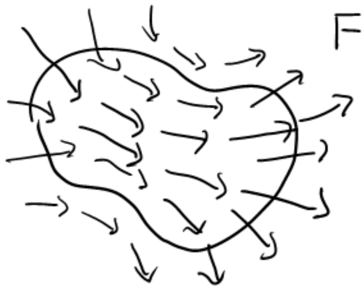
$$\vec{n}(t) = \langle h(t), -g(t)\rangle \frac{1}{|V(t)|} = \left\langle \frac{dy}{dt}, -\frac{dx}{dt}\right\rangle \frac{1}{|V(t)|}$$

(because $|\vec{n}| = 1$ and $\vec{n}\cdot T = 0$.)

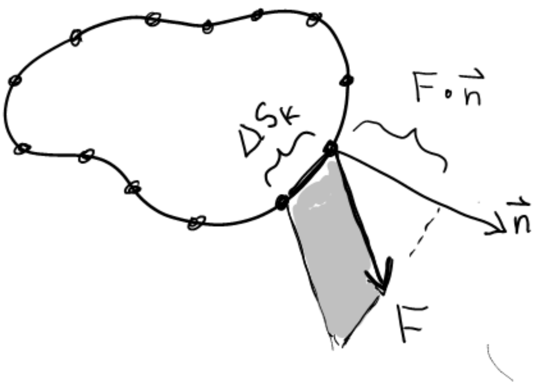


Topic 2 Outward Flux

Think of F as representing the velocity of a fluid flowing on the plane. The outward flux is the net flow out of the region (in, say, square units per second). Here's how to compute outward flux:



Divide C into subintervals, lengths $\Delta S_1 \cdots \Delta S_n$. Outward flow through segment ΔS_K is approx. (Area of shaded region) = (height)(base) $= F\cdot\vec{n}\,\Delta S_K$ square units / second.



Therefore: Flux = Net flow out of C

$$\approx \sum_{K=1}^{n} F\cdot\vec{n}\,\Delta S_K$$

$$\text{Flux} = \text{Net flow out of } C = \lim_{\|P\|\to 0} \sum_{K=1}^{n} F\cdot\vec{n}\,\Delta S_K$$

$$= \boxed{\oint_C F\cdot\vec{n}\,ds} = \oint_C \underbrace{\langle M, N\rangle}_{F} \cdot \underbrace{\left\langle \frac{dy}{dt}, -\frac{dx}{dt}\right\rangle \frac{1}{|V(t)|}}_{\vec{n}} \underbrace{|V(t)|\,dt}_{ds}$$

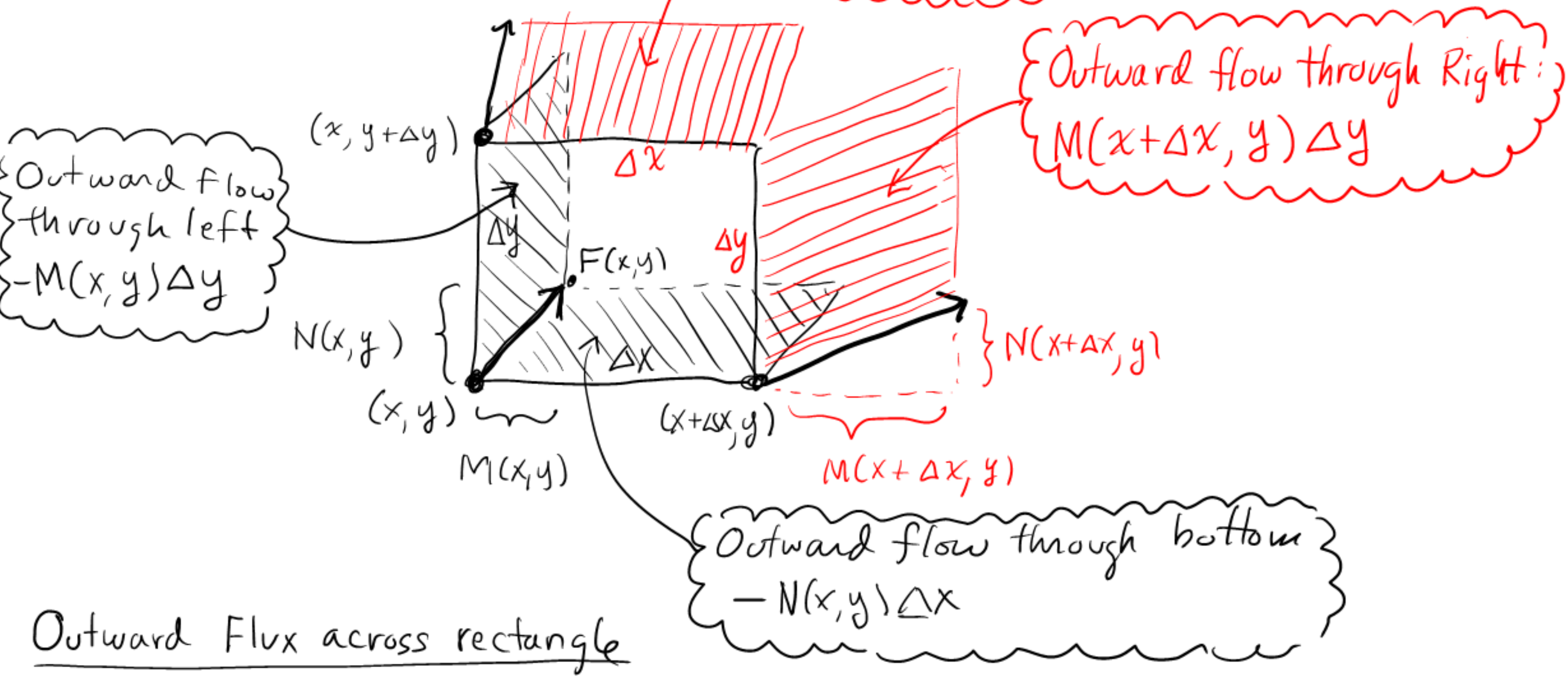
$$= \oint_C \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right) dt = \boxed{\oint_C M\,dy - N\,dx.}$$

Thus left side in Green's Theorem is outward flux through C.

Topic 3 Another View of outward flux.

Consider a small rectangle enclosing a region of the plane, and the vector field F, as above, representing a fluid's velocity.

Approximations to outward flux across four sides:



Outward Flux across rectangle

$$\approx M(x+\Delta x, y)\Delta y - M(x,y)\Delta y + N(x, y+\Delta y)\Delta x - N(x,y)\Delta x$$

$$= \frac{M(x+\Delta x, y) - M(x,y)}{\Delta x}\Delta x\,\Delta y + \frac{N(x, y+\Delta y) - N(x,y)}{\Delta y}\Delta x\,\Delta y$$

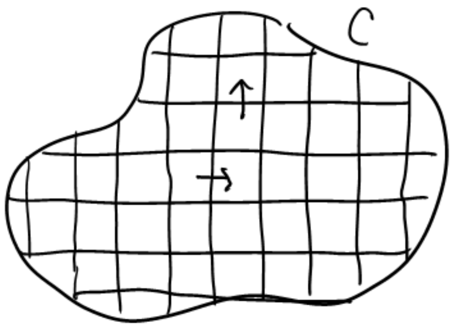
$$\approx \frac{\partial M}{\partial x}\Delta x\,\Delta y + \frac{\partial N}{\partial y}\Delta x\,\Delta y = \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}\Delta x\,\Delta y.$$

Called the *divergence* of F. Measures outward flux. More on this later.

Now chop R into small rectangles. Then:

Flux across C $\approx \sum \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial N}\right)\Delta x_k\,\Delta y_k$

(Note contribution to flux common side of adjacent rectangles is zero. One positive – other negative)



$$\boxed{\text{Thus flux across } C = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

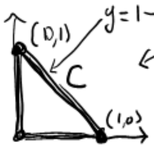
We have now computed the same thing – outward flux – in two ways:

$$\text{Outward flux} = \oint_C F \cdot \vec{n}\, ds = \oint M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy .$$

This is Green's Theorem.

Example 16.4 (21) Green's Theorem can be used to evaluate certain line integrals.

Find $\oint_C y^2 dx + x^2 dy$



computing this directly would involve doing three seperate line integrals – one for each of the three sides of the triangle

$$= \oint_C \underset{\substack{\uparrow \\ M}}{x^2}\, dy - \underset{N}{\underbrace{(-y^2)}}\, dx = \iint_R (2x - 2y)\, dx\, dy = \int_0^1 \int_0^{1-x} (2x - 2y)\, dy\, dx = \int_0^1 \left[2xy - y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left(2x(1-x) - (1-x)^2\right) dx = \int_0^1 (2x - 2x^2 - 1 + 2x - x^2)\, dx = \int_0^1 (4x - 1 - 3x^2)\, dx = \left[2x^2 - x - x^3\right]_0^1 = \boxed{0}$$

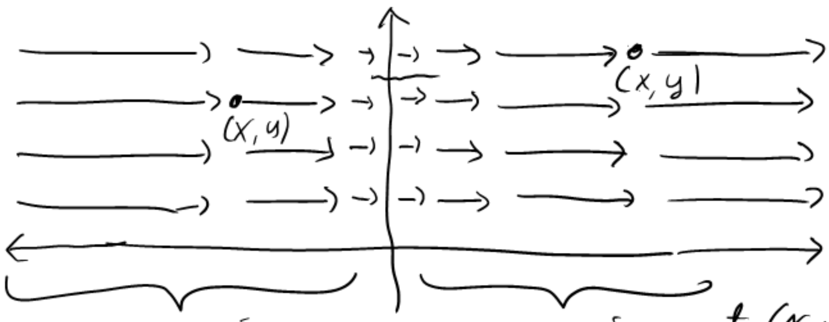
$$\underline{\text{Divergence of } F} = \text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \left(\begin{array}{l} \text{Measure of outward} \\ \text{flux through small} \\ \text{region at } (x,y) \end{array} \right)$$

div $F > 0$ (positive flux) = expansion

div $F < 0$ (negative flux) = compression

div $F = 0$ no compression / expansion — $\left\{ \begin{array}{l} F \text{ represents flow} \\ \text{of indecompressible} \\ \text{fluid} \end{array} \right.$

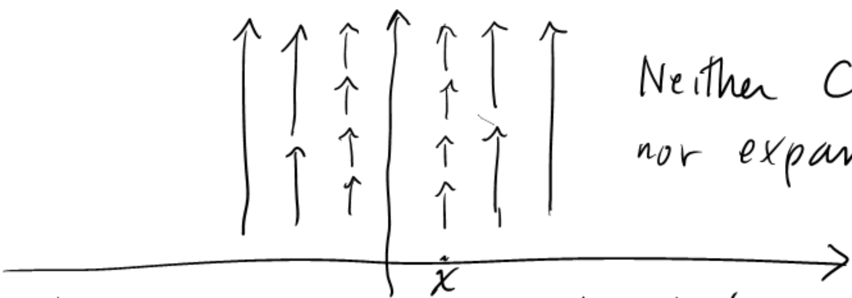
Example $F = \langle x^2, 0 \rangle$ $\quad \text{div } F = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} = 2x$



compression at (x,y)
div $F = 2x < 0$

expansion at (x,y)
div $F = 2x > 0$

Example $F = \langle 0, x^2 \rangle$ $\quad \text{div } F = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} = 0$



Neither Compression nor expansion

See other examples in text!