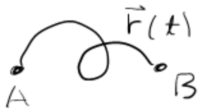
Section 16.3 Path Independence, Conservative Fields, Potential Functions

Fundamental Theorem of Calculus:

Given interval 

, have $\int_a^b f'(x)\,dx = f(b) - f(a)$

Todays Goal

Given curve 

, have $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$

Reaching this goal involves several new ideas

Path Independence

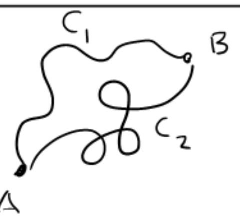
Given two paths joining A to B 

usually $\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\cdot d\vec{r}$

But under just the right circumstances, these integrals are <u>equal</u>.

Suppose for some vector field F it happens that

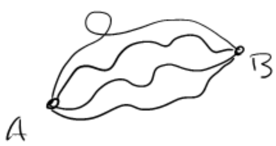
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$



whenever C and C_2 are two curves joining that that begin at the same point and end at the same point.

Then $\int_C \vec{F}\cdot d\vec{r}$ is said to be <u>path independent</u>.

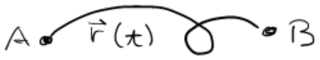
A vector field $\vec{F}$ having this property is called a <u>conservative</u> v.f.



<u>Note</u>. For a conservative v.f. $\int_C \vec{F}\cdot d\vec{r}$ has the same value for all curves C joining A to B.

Question What vector fields are conservative?

Theorem 1 Suppose $\vec{F} = \nabla f$ for some function $f(x,y,z)$ (or $f(x,y)$).

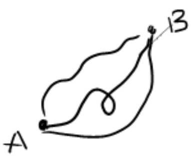
Then for any curve C 

in the domain of f,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

i.e. $$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Note: $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$ means that the integral depends <u>only</u> on the endpoints and not the curve itself



Thus $\vec{F}$ is a conservative field: The integral equals $f(B) - f(A)$ for <u>any</u> curve joining A to B.

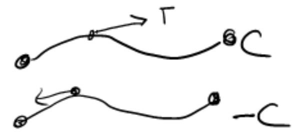
Thus $(\vec{F} = \nabla f) \Longrightarrow (\vec{F} \text{ is conservative})$

Theorem 2 $\left(\vec{F} = \nabla f \text{ for some function } f\right) \iff \left(\vec{F} \text{ is conservative}\right)$

(The proof of this is in the text)

Let $-C$ denote curve C traversed backwards

Then $\int_C F \cdot dr = \int_C F \cdot T\,ds = -\int_{-C} F \cdot T\,ds = -\int_{-C} F \cdot dr$



Notice that if F is conservative then for any closed curve C we have $\int_C F \cdot d\vec{r} = f(A) - f(A) = 0$

Conversely, suppose $\int_C F \cdot dr = 0$ for every closed C

Take two paths from A to B. Then $C_1 \cup -C_2$ is closed

Thus $0 = \int_{C_1 \cup -C_2} F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr \Rightarrow \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$

This means F is conservative.

Conclusion

Theorem 3 $(F \text{ conservative}) \iff \left(\int_C F \cdot dr = 0 \text{ for any closed curve } C\right)$

Definition If $\vec{F} = \nabla f$, then f is called a potential function for $\vec{F}$

Example: $\vec{F}(x, y, z) = \langle zy\cos(xy),\ zx\cos(xy),\ \sin(xy) \rangle$

Potential function for $\vec{F}$ is $f(x,y,z) = z\sin(xy)$ because $\vec{F} = \nabla f$

Note: Another potential function is $f(x,y,z) = z\sin(xy) + 5$

Compare:

F.T.C $\quad \int_a^b F(t)\,dt = f(b) - f(a)$ where $f'(t) = F(t)$ (f is antiderivative of F)

Theorem 1 $\quad \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$ where $\nabla f = F$ (f is potential function of F)

Therefore potential functions are "antiderivatives" of vector fields.

The equation $\int_C F \cdot d\vec{r} = f(B) - f(A)$ when $\nabla f = F$ gives a quick way to compute the integral provided we can find the potential function F. NEXT TIME Computing Potential Functions.