Fundamental Theorem of Calculus:

Given interval \( a \to b \), have
\[
\int_a^b f'(x) \, dx = f(b) - f(a)
\]

Today's Goal:

Given curve \( \mathbf{F}(t) \) have
\[
\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)
\]

Reaching this goal involves several new ideas.

Path Independence:

Given two paths joining \( A \) to \( B \), usually
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

But under just the right circumstances, these integrals are equal.

Suppose for some vector field \( \mathbf{F} \) it happens that
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]
whenever \( C_1 \) and \( C_2 \) are two curves joining \( A \) to \( B \) that begin at the same point and end at the same point.

Then \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is said to be path independent.

A vector field \( \mathbf{F} \) having this property is called a conservative v.f.

Note. For a conservative v.f., \( \int_C \mathbf{F} \cdot d\mathbf{r} \) has the same value for all curves \( C \) joining \( A \) to \( B \).

Question: What vector fields are conservative?

Theorem 1: Suppose \( \mathbf{F} = \nabla f \) for some function \( f(x,y,z) \) (or \( f(x,y) \)).

Then for any curve \( C \) from \( A \) to \( B \) in the domain of \( f \),
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)
\]

i.e. \( \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) \)

Note: \( \int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) \) means that the integral depends only on the endpoints and not the curve itself.

Thus \( \mathbf{F} \) is a conservative field: The integral equals \( f(B) - f(A) \) for any curve joining \( A \) to \( B \).

Thus \( (\mathbf{F} = \nabla f) \implies (\mathbf{F} \text{ is conservative}) \)
Theorem 2 \((\mathbf{F} = \nabla f \text{ for some function } f) \iff (\mathbf{F} \text{ is conservative})\)

(The proof of this is in the text)

Let \(-C\) denote curve \(C\) traversed backwards.
Then \(\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}\)

Notice that if \(\mathbf{F}\) is conservative then for any closed curve \(C\) we have \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(A) - f(A) = 0\).

Conversely, suppose \(\int_C \mathbf{F} \cdot d\mathbf{r} = 0\) for every closed \(C\).
Take two paths from \(A\) to \(B\). Then \(C_1 \cup C_2\) is closed.
Thus \(0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\).
This means \(\mathbf{F}\) is conservative.

Conclusion

Theorem 3 \((\mathbf{F} \text{ conservative}) \iff (\int_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed curve } C)\)

Definition If \(\mathbf{F} = \nabla f\), then \(f\) is called a potential function for \(\mathbf{F}\).

Example: \(\mathbf{F}(x, y, z) = \langle zy \cos(xy), xz \cos(xy), \sin(xy) \rangle\)
Potential function for \(\mathbf{F}\) is \(f(x, y, z) = z \sin(xy)\) because \(\mathbf{F} = \nabla f\).
Note: Another potential function is \(f(x, y, z) = z \sin(xy) + 5\).

Compare:
\[
\int_a^b F(t) \, dt = F(b) - F(a) \quad \text{where } F(t) = F(x) \quad (F \text{ is antiderivative of } F)
\]

Theorem 1 \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) \quad \text{where } \nabla f = F \quad (F \text{ is potential function of } F)\)

Therefore potential functions are "antiderivatives" of vector fields.

The equation \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) \quad \text{where } \nabla f = F\) gives a quick way to compute the integral provided we can find the potential function \(F\). Next time Computing Potential Functions.