Goal: Determine the center of gravity of a flat plate, that is the point at which it would balance on the tip of a pencil. Also compute the center of gravity of a 3-D solid.

Mass of a Plate
Suppose a flat plate $R$ has density $\delta(x, y)$ grams per square unit at point $(x, y)$.

What is total mass of plate?

Divide $R$ up into rectangles, $R_1, R_2, \ldots, R_n$. In each rectangle $R_k$, put a sample point $(x_k, y_k)$.

\[
\text{(mass of rectangle } k \text{)} = (\text{density})(\text{area}) \approx \delta(x_k, y_k) \Delta x_k \Delta y_k
\]

\[
\text{Mass} \approx \sum_{k=1}^{n} \delta(x_k, y_k) \Delta x_k \Delta y_k
\]

\[
\text{Mass} = \lim_{|P| \to 0} \sum_{k=1}^{n} \delta(x_k, y_k) \Delta x_k \Delta y_k = \iint_{R} \delta(x, y) \, dA
\]

Now, let's turn our attention to centers of mass.

Masses $m_1$ and $m_2$ balance if $m_1d_1 = m_2d_2$.

Masses $m_1$ and $m_2$ on number line balance at $\bar{x}$ if

\[
(x_1 - \bar{x})m_1 + (x_2 - \bar{x})m_2 = 0
\]

i.e.

\[
\sum_{k=1}^{2} (x_k - \bar{x})m_k = 0
\]

Masses $m_1, m_2, \ldots, m_n$ balance at $\bar{x}$ if

\[
\sum_{k=1}^{n} (x_k - \bar{x})m_k = 0
\]
Now imagine masses \( m_1, m_2, \ldots, m_n \) are distributed on the \( xy \)-plane and any given mass \( m_k \) is at the point \( (x_k, y_k) \).

System will balance on the line \( x = \overline{x} \) provided that
\[
\sum_{k=1}^{n} (x_k - \overline{x}) m_k = 0
\]

Next, we are going to compute the vertical line \( x = \overline{x} \) that a region \( R \) balances on.

Consider region \( R \) whose density at \( (x, y) \) is \( s(x, y) \).

Divide into \( n \) rectangles.

Put sample point \( (x_k, y_k) \) in \( R_k \).

Then \( R_k \) has mass \( s(x_k, y_k) \Delta x_k \Delta y_k \).

Region will balance at line \( x = \overline{x} \) provided that
\[
\sum_{k=1}^{n} (x_k - \overline{x}) s(x_k, y_k) \Delta x_k \Delta y_k = 0
\]

\[
\iint_{R} (x - \overline{x}) s(x, y) \, dA = 0
\]

\[
\iint_{R} x s(x, y) \, dA - \iint_{R} \overline{x} s(x, y) \, dA = 0
\]

\[
\overline{x} = \frac{\iint_{R} x s(x, y) \, dA}{\iint_{R} s(x, y) \, dA} = \frac{My}{M}
\]

**Similar Calculations**

on the horizontal line \( y = \overline{y} \), where
\[
\overline{y} = \frac{\iint_{R} y s(x, y) \, dA}{\iint_{R} s(x, y) \, dA} = \frac{Mx}{M}
\]

Region balances at the intersection of these lines, i.e., at point \( (\overline{x}, \overline{y}) \).
Summary

Suppose a 2-D plate has density \( s(x,y) \) at \((x,y)\).
Then:

\[
\text{Mass} = M = \iint \limits_{R} s(x,y) \, dA
\]

"First moments"
\[
\begin{align*}
M_y &= \iiint \limits_{R} x \, s(x,y) \, dA \\
M_x &= \iiint \limits_{R} y \, s(x,y) \, dA
\end{align*}
\]

Center of mass, \((\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right)\)

Suppose a 3-D solid has density \( s(x,y,z) \) at \((x,y,z)\).
Then

\[
\text{Mass} = M = \iiint \limits_{D} s(x,y,z) \, dV
\]

First moments
\[
\begin{align*}
M_{yz} &= \iiint \limits_{D} x \, s(x,y,z) \, dV \\
M_{xz} &= \iiint \limits_{D} y \, s(x,y,z) \, dV \\
M_{yz} &= \iiint \limits_{D} z \, s(x,y,z) \, dV
\end{align*}
\]

Center of mass: \((x, y, z) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{yz}}{M} \right)\)

Note: Can ignore material on moments of inertia
Example

Find the center of mass of the region which has uniform density of 8 grams per square foot.

\[
\text{Mass } M = \iiint_R s \, dA = \int_0^1 \int_0^{\sqrt{x}} s \, dy \, dx = \int_0^1 [sy]_{\sqrt{x}}^1 \, dx
\]

\[
= \int_0^1 8 \sqrt{x} \, dx = \left[ 8 \frac{x^{3/2}}{3/2} \right]_0^1 = \left[ \frac{16}{3} \right]_0^1 = \frac{16}{3} \text{ grams}
\]

\[
M_y = \iint_R sx \, dA = \int_0^1 \int_0^{\sqrt{x}} sx \, dy \, dx = \int_0^1 [sxy]_{\sqrt{x}}^1 \, dx
\]

\[
= \int_0^1 sx \sqrt{x} \, dx = \int_0^1 8x^{3/2} \, dx = \left[ 8 \frac{x^{5/2}}{5/2} \right]_0^1 = \left[ \frac{16}{5} \right]_0^1 = \frac{16}{5}
\]

\[
M_x = \iint_R sy \, dA = \int_0^1 \int_0^{\sqrt{x}} sy \, dy \, dx = \int_0^1 [s y^2]_{\sqrt{x}}^1 \, dx
\]

\[
= \int_0^1 s \frac{x}{2} \, dx = \left[ s \frac{x^2}{4} \right]_0^1 = \left[ s \frac{1}{4} \right]_0^1 = \frac{s}{4}
\]

Center of mass: \((\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{\frac{16}{5}}{\frac{16}{3}}, \frac{\frac{s}{4}}{\frac{16}{3}} \right) = \left( \frac{3}{5}, \frac{3}{8} \right) = (\frac{3}{5}, \frac{3}{8})
\]

\[
= \left( \frac{3}{5}, \frac{3}{8} \right) = \left( 0.6, 0.375 \right)
\]
Example (Time permitting)

Density \( S(x, y) = 4x + 2y + 2 \) grams/sq unit at point \((x, y)\). Find mass \( M \) and center of mass \((\bar{x}, \bar{y})\).

\[
M = \iint_R (4x + 2y + 2) \, dA = \int_0^2 \int_0^{2x} (4x + 2y + 2) \, dy \, dx
\]

\[
= \int_0^2 \left[ 4xy + y^2 + 2y \right]_0^{2x} \, dx = \int_0^2 (8x^2 + 4x) \, dx
\]

\[
= \left[ 4x^3 + 2x^2 \right]_0^2 = 32 + 8 = \boxed{40 \text{ grams}}
\]

\[
M_y = \iint_R x(4x + 2y + 2) \, dA = \int_0^2 \int_0^{2x} 4x^2 + 2xy + 2x \, dy \, dx
\]

\[
= \int_0^2 \left[ 4x^2y + xy^2 + 2xy \right]_0^{2x} \, dx = \int_0^2 (8x^3 + 4x^2) \, dx
\]

\[
= \left[ \frac{2}{3} x^3 + 4x^2 \right]_0^2 = \frac{16}{3} + \frac{4}{3} \cdot 8 = 14 \cdot \frac{32}{3} = \boxed{\frac{176}{3}}
\]

\[
M_x = \iint_R y(4x + 2y + 2) \, dA = \int_0^2 \int_0^{2x} 4xy + 2y^2 + 2y \, dx \, dy
\]

\[
= \int_0^2 \left[ 2xy^2 + \frac{2}{3} y^3 + y^2 \right]_0^{2x} \, dx = \int_0^2 (8x^3 + 16) \, dx
\]

\[
= \int_0^2 \frac{40}{3} x^3 + 4x^2 \, dx = \left[ \frac{10}{3} x^4 + \frac{4}{3} x^3 \right]_0^2
\]

\[
= \frac{160}{3} + \frac{32}{3} = \boxed{\frac{192}{3}}
\]

Center of mass: \( \left( \frac{M_x}{M}, \frac{M_y}{M} \right) = \left( \frac{176}{40}, \frac{192}{40} \right) \)

\[
= \left( \frac{22}{15}, \frac{8}{5} \right)
\]