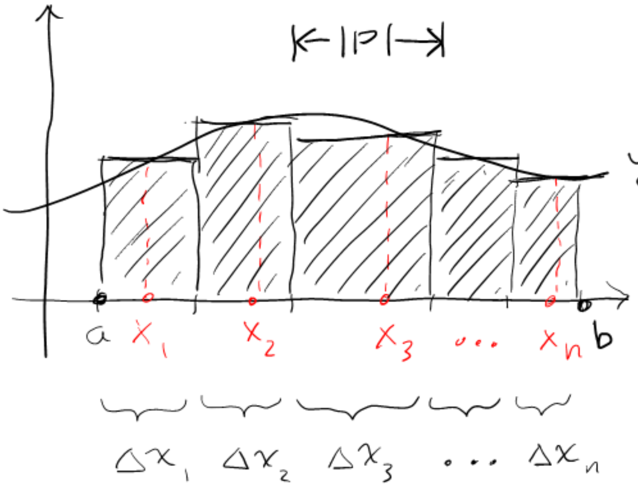
Chapter 15 Multiple Integrals

Section 15.1 Double Integrals Over Rectangles

Recall the setup for the definition of the definite integral of $f(x)$ over the interval $[a,b]$:



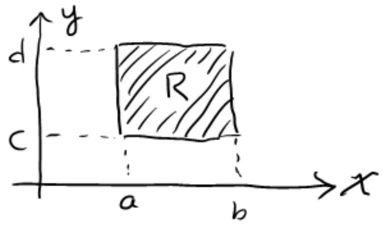
- **Norm** of the partition P is $|P| =$ largest Δx_k.
- Number of rectangles is n
- As $|P| \to 0$, $n \to \infty$
- Each x_k is a "sample point"
- Riemann sum: $\sum_{k=1}^{n} f(x_k)\Delta x_k$

Definite Integral $\displaystyle\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$

$= \left(\begin{array}{l}\text{area under curve if} \\ f(x) \geq 0 \text{ for all } a \leq x \leq b\end{array}\right)$

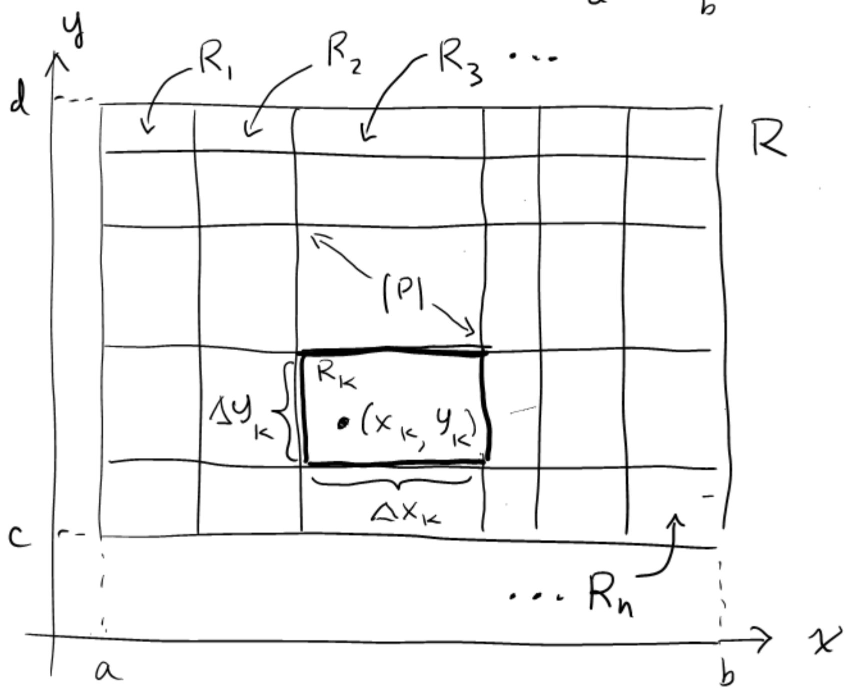
Fund. Theorem of Calc: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where $F' = f$.

Now we will adapt this from $f(x)$ to $f(x,y)$. Instead of an interval $[a,b]$ for inputs x, there is a rectangle R for inputs (x,y).

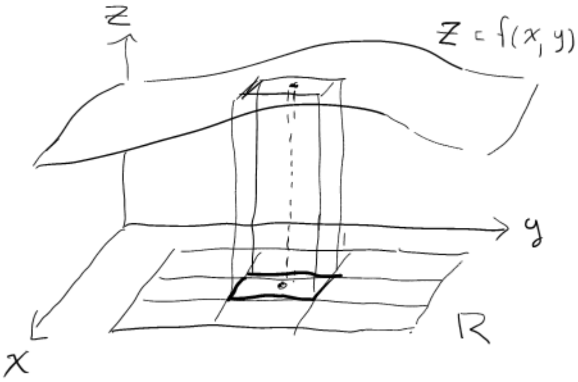


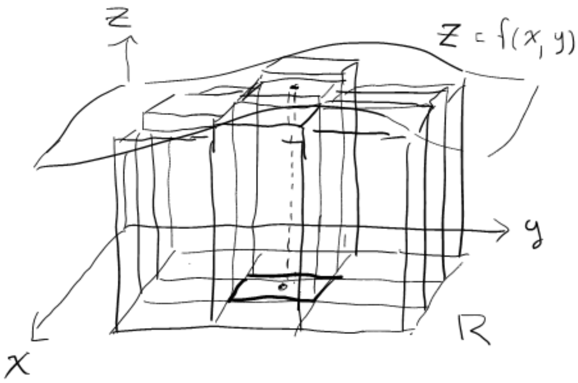
Partition R into n smaller rectangles $R_1, R_2, \ldots R_n$

- R_k has dimensions $\Delta x_k \times \Delta y_k$ and area $A_k = \Delta x_k \Delta y_k$
- $|P| =$ length of longest diagonal
- As $|P| \to 0$, $n \to \infty$
- Inside each R_k is a sample point (x_k, y_k)



- Riemann sum: $\displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$





$$f(x_k, y_k)\,\Delta A_k = f(x_k, y_k)\,\Delta x_k\,\Delta y_k$$
$$= (\text{height})(\text{length})(\text{width})$$
$$= \text{Volume of box}$$

$$\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$
$$= (\text{sum of box volumes})$$
$$\approx \text{volume under graph}$$

Note: Sum of box volumes can be negative if there are negative $f(x_k, y_k)$.

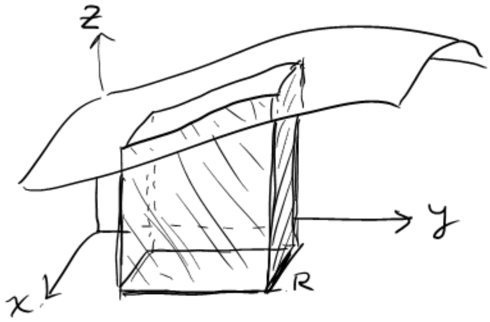
Definition The *definite integral* of $f(x,y)$ over rectangle R is number

$$\iint_R f(x,y)\,dA = \lim_{|P|\to 0}\left(\sum_{k=1}^{n} f(x_k, y_k)\,dA_k\right)$$

provided this limit exists. If it does, we say that $f(x,y)$ is *integrable* over the region R.

Note: If $f(x,y) > 0$ on R then

$$\iint_R f(x,y)\,dA = \begin{pmatrix}\text{Volume under graph} \\ \text{of } z = f(x,y) \text{ and} \\ \text{over rectangle } R\end{pmatrix}$$



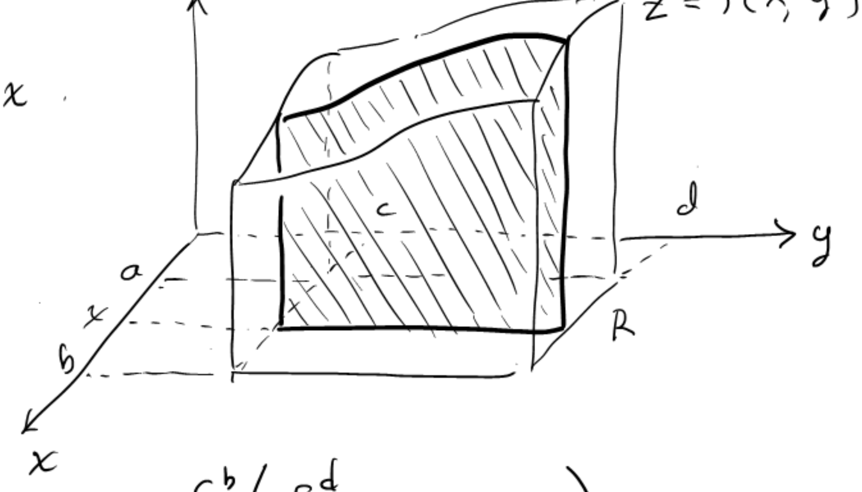
Theorem If $f(x,y)$ is continuous on R then it is integrable on R

Computing Double Integrals

Area of cross-section at x.

$$A(x) = \int_c^d f(x,y)\,dy$$

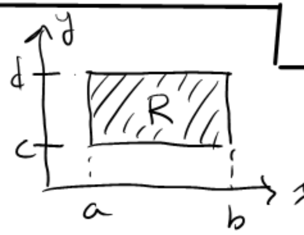
think of x as constant



Volume under $z = f(x,y)$ is

$$\iint_R f(x,y)\,dA = \int_a^b A(x)\,dx = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

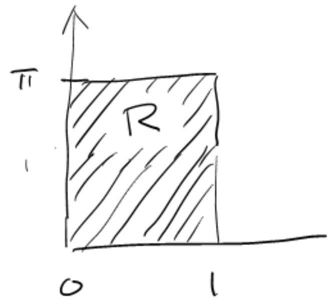
From this computation, we get:



Theorem 1 Suppose $f(x,y)$ is continuous on rectangle R:

Then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$

Example $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le \pi\}$



$$\iint_R x\cos(xy)\,dA = \int_0^1\int_0^\pi x\cos(xy)\,dy\,dx$$

$$= \int_0^1 \left[x \cdot \frac{1}{x}\sin(xy)\right]_0^\pi dx$$

$$= \int_0^1 \left[\sin(xy)\right]_0^\pi dx = \int_0^1 \left(\sin(x\pi) - \sin(x\cdot 0)\right) dx$$

$$= \int_0^1 \sin(\pi x)\,dx = \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1$$

$$= -\frac{1}{\pi}\cos\pi - \left(-\frac{1}{\pi}\cos 0\right) = \frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}$$

On the other hand, what if we tried...

$$\iint_R x\cos(xy)\,dA = \int_0^\pi\int_0^1 x\cos(xy)\,dx\,dy$$

$$= \int_0^\pi \left[\frac{x}{y}\sin(xy) + \frac{1}{y^2}\cos(xy)\right]_0^1 dy$$

$$= \int_0^\pi \left(\frac{1}{y}\sin y + \frac{1}{y^2}\cos y - \frac{1}{y^2}\right) dy$$

= TOUGH INTEGRAL

Integration by parts:

$\int \cos(xy)\, x\,dx$

$u = x \qquad du = dx$

$dv = \cos(xy)\,dx \qquad v = \frac{1}{y}\sin(xy)$

$\int \cos(xy)\,x\,dx = uv - \int v\,du$

$= \frac{x}{y}\sin(xy) - \int \frac{1}{y}\sin(xy)\,dx$

$= \frac{x}{y}\sin(xy) + \frac{1}{y^2}\cos(xy)$

Moral: Although $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$ sometimes one double integral is easier than the other!